

# CR RIGIDITY OF PSEUDO HARMONIC MAPS AND PSEUDO BIHARMONIC MAPS

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**ABSTRACT.** The  $CR$  analogue of B.-Y. Chen's conjecture on pseudo biharmonic maps will be shown. Pseudo biharmonic, but not pseudo harmonic, isometric immersions with parallel pseudo mean curvature vector fields, will be characterized.

## 1. INTRODUCTION

Harmonic maps play a central role in geometry; they are critical points of the energy functional  $E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$  for smooth maps  $\varphi$  of  $(M, g)$  into  $(N, h)$ . The Euler-Lagrange equations are given by the vanishing of the tension field  $\tau(\varphi)$ . In 1983, Eells and Lemaire [12] extended the notion of harmonic map to biharmonic map, which are critical points of the bienergy functional  $E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g$ . After Jiang [20] studied the first and second variation formulas of  $E_2$ , extensive studies in this area have been done (for instance, see [6], [22], [25], [16], [17], [19]). Every harmonic map is always biharmonic by definition. Chen raised ([7]) famous Chen's conjecture and later, Caddeo, Montaldo, Piu and Oniciuc raised ([6]) the generalized Chen's conjecture.

**B.-Y. Chen's conjecture:**

*Every biharmonic submanifold of the Euclidean space  $\mathbb{R}^n$  must be harmonic (minimal).*

**The generalized B.-Y. Chen's conjecture:**

*Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic (minimal).*

For the generalized Chen's conjecture, Ou and Tang gave ([30]) a counter example in a Riemannian manifold of negative curvature. For

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Chen's conjecture, some affirmative answers were known for surfaces in the three dimensional Euclidean space ([7]), and hypersurfaces of the four dimensional Euclidean space ([15], [9]). Akutagawa and Maeta showed ([1]) that any complete regular biharmonic submanifold of the Euclidean space  $\mathbb{R}^n$  is harmonic (minimal).

To the generalized Chen's conjecture, we showed ([28]) that: for a complete Riemannian manifold  $(M, g)$ , a Riemannian manifold  $(N, h)$  of non-positive curvature, then, every biharmonic map  $\varphi : (M, g) \rightarrow (N, h)$  with finite energy and finite bienergy is harmonic. In the case  $\text{Vol}(M, g) = \infty$ , every biharmonic map  $\varphi : (M, g) \rightarrow (N, h)$  with finite bienergy is harmonic. This gave ([26], [27], [28]) affirmative answers to the generalized Chen's conjecture under the  $L^2$ -condition and the completeness of  $(M, g)$ .

In 1970's, Chern and Moser initiated ([8]) the geometry and analysis of strictly convex  $CR$  manifolds, and many mathematicians works on  $CR$  manifolds (cf. [11]). Recently, Barletta, Dragomir and Urakawa gave ([5]) the notion of pseudo harmonic map, and also Dragomir and Montaldo settled ([10]) the one of pseudo biharmonic map.

In this paper, we raise

**The  $CR$  analogue of the generalized Chen's conjecture:**

*Let  $(M, g_\theta)$  be a complete strictly pseudoconvex  $CR$  manifold, and assume that  $(N, h)$  is a Riemannian manifold of non-positive curvature. Then, every pseudo biharmonic isometric immersion  $\varphi : (M, g_\theta) \rightarrow (N, h)$  must be pseudo harmonic.*

We will show this conjecture holds under some  $L^2$  condition on a complete strongly pseudoconvex  $CR$  manifold (cf. Theorem 3.2), and will give characterization theorems on pseudo biharmonic immersions from  $CR$  manifolds into the unit sphere or the complex projective space (cf. Theorems 6.2 and 7.1). More precisely, we will show

**Theorem 1.1.** *(cf. Theorem 3.2) Let  $\varphi$  be a pseudo biharmonic map of a complete  $CR$  manifold  $(M, g_\theta)$  into a Riemannian manifold  $(N, h)$  of non-positive curvature. Then,*

*If the pseudo energy  $E_b(\varphi)$  and the pseudo bienergy  $E_{b,2}(\varphi)$  are finite, then  $\varphi$  is pseudo harmonic.*

For isometric immersions of a  $CR$  manifold  $(M^{2n+1}, g_\theta)$  into the unit sphere  $S^{2n+2}(1)$  of curvature 1, we have

**Theorem 1.2.** *(cf. Theorem 6.2) For such immersion, assume that the pseudo mean curvature is parallel, but not pseudo harmonic.*

Then,  $\varphi$  is pseudo biharmonic if and only if the restriction of the second fundamental form  $B_\varphi$  to the holomorphic subspace  $H_x(M)$  of  $T_x M$  ( $x \in M$ ) satisfies that

$$\| B_\varphi|_{H(M) \times H(M)} \|^2 = 2n.$$

For isometric immersions of a CR manifold  $(M^{2n+1}, g_\theta)$  into the complex projective space  $(\mathbb{P}^{n+1}(c), h, J)$  of holomorphic sectional curvature  $c > 0$ , we have

**Theorem 1.3.** (cf. Theorem 7.1) *For such immersion, assume that the pseudo mean curvature is parallel, but not pseudo harmonic. Then,  $\varphi$  is pseudo biharmonic if and only if one of the following holds:*

(1)  $J(d\varphi(T))$  is tangent to  $\varphi(M)$  and

$$\| B_\varphi|_{H(M) \times H(M)} \|^2 = \frac{c}{4}(2n + 3).$$

(2)  $J(d\varphi(T))$  is normal to  $\varphi(M)$  and

$$\| B_\varphi|_{H(M) \times H(M)} \|^2 = \frac{c}{4}(2n) = \frac{n}{2} c.$$

Here,  $T$  is the characterstic vector field of  $(M, g_\theta)$ ,  $H_x(M) \oplus \mathbb{R}T_x = T_x(M)$ , and  $B_\varphi|_{H(M) \times H(M)}$  is the restriction of the second fundamental form  $B_\varphi$  to  $H_x(M)$  ( $x \in M$ ).

Several examples of pseudo biharmonic immersions of  $(M, g_\theta)$  into the unit sphere or complex projective space will be given.

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## 2. PRELIMINARIES

**2.1.** We prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map  $\varphi : (M, g) \rightarrow (N, h)$ , of a compact Riemannian manifold  $(M, g)$  into another Riemannian manifold  $(N, h)$ , which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where  $e(\varphi) := \frac{1}{2}|d\varphi|^2$  is called the energy density of  $\varphi$ . That is, for any variation  $\{\varphi_t\}$  of  $\varphi$  with  $\varphi_0 = \varphi$ ,

$$\left. \frac{d}{dt} \right|_{t=0} E(\varphi_t) = - \int_M h(\tau(\varphi), V) v_g = 0, \quad (2.1)$$

where  $V \in \Gamma(\varphi^{-1}TN)$  is a variation vector field along  $\varphi$  which is given by  $V(x) = \left. \frac{d}{dt} \right|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N$ , ( $x \in M$ ), and the *tension field* is given by  $\tau(\varphi) = \sum_{i=1}^m B_\varphi(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$ , where  $\{e_i\}_{i=1}^m$  is a locally defined orthonormal frame field on  $(M, g)$ , and  $B_\varphi$  is the second fundamental form of  $\varphi$  defined by

$$\begin{aligned} B_\varphi(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X^g Y), \end{aligned} \quad (2.2)$$

for all vector fields  $X, Y \in \mathfrak{X}(M)$ . Here,  $\nabla^g$ , and  $\nabla^h$ , are Levi-Civita connections on  $TM$ ,  $TN$  of  $(M, g)$ ,  $(N, h)$ , respectively, and  $\overline{\nabla}$ , and  $\widetilde{\nabla}$  are the induced ones on  $\varphi^{-1}TN$ , and  $T^*M \otimes \varphi^{-1}TN$ , respectively. By (2.1),  $\varphi$  is *harmonic* if and only if  $\tau(\varphi) = 0$ .

The second variation formula is given as follows. Assume that  $\varphi$  is harmonic. Then,

$$\left. \frac{d^2}{dt^2} \right|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \quad (2.3)$$

where  $J$  is an elliptic differential operator, called the *Jacobi operator* acting on  $\Gamma(\varphi^{-1}TN)$  given by

$$J(V) = \overline{\Delta}V - \mathcal{R}(V), \quad (2.4)$$

where  $\overline{\Delta}V = \overline{\nabla}^* \overline{\nabla}V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i}^g e_i} V\}$  is the *rough Laplacian* and  $\mathcal{R}$  is a linear operator on  $\Gamma(\varphi^{-1}TN)$  given by  $\mathcal{R}(V) = \sum_{i=1}^m R^h(V, d\varphi(e_i))d\varphi(e_i)$ , and  $R^h$  is the curvature tensor of  $(N, h)$  given by  $R^h(U, V) = \nabla_U^h \nabla_V^h - \nabla_V^h \nabla_U^h - \nabla_{[U, V]}^h$  for  $U, V \in \mathfrak{X}(N)$ .

J. Eells and L. Lemaire [12] proposed polyharmonic ( $k$ -harmonic) maps and Jiang [20] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \quad (2.5)$$

where  $|V|^2 = h(V, V)$ ,  $V \in \Gamma(\varphi^{-1}TN)$ .

The first variation formula of the bienergy functional is given by

$$\left. \frac{d}{dt} \right|_{t=0} E_2(\varphi_t) = - \int_M h(\tau_2(\varphi), V) v_g. \quad (2.6)$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \quad (2.7)$$

which is called the *bitension field* of  $\varphi$ , and  $J$  is given in (2.4).

A smooth map  $\varphi$  of  $(M, g)$  into  $(N, h)$  is said to be *biharmonic* if  $\tau_2(\varphi) = 0$ . By definition, every harmonic map is biharmonic. For an isometric immersion, it is minimal if and only if it is harmonic.

**2.2.** Following Dragomir and Montaldo [10], and also Barletta, Dragomir and Urakawa [5], we will prepare the materials on pseudo harmonic maps and pseudo biharmonic maps.

Let  $M$  be a strictly pseudoconvex  $CR$  manifold of  $(2n+1)$ -dimension,  $T$ , the characteristic vector field on  $M$ ,  $J$  is the complex structure of the subspace  $H_x(M)$  of  $T_x(M)$  ( $x \in M$ ), and  $g_\theta$ , the Webster Riemannian metric on  $M$  defined for  $X, Y \in H(M)$  by

$$g_\theta(X, Y) = (d\theta)(X, JY), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1.$$

Let us recall for a  $C^\infty$  map  $\varphi$  of  $(M, g_\theta)$  into another Riemannian manifold  $(N, h)$ , the *pseudo energy*  $E_b(\varphi)$  is defined ([5]) by

$$E_b(\varphi) = \frac{1}{2} \int_M \sum_{i=1}^{2n} (\varphi^* h)(X_i, X_i) \theta \wedge (d\theta)^n, \quad (2.8)$$

where  $\{X_i\}_{i=1}^{2n}$  is an orthonormal frame field on  $(H(M), g_\theta)$ . Then, the first variational formula of  $E_b(\varphi)$  is as follows ([5]). For every variation  $\{\varphi_t\}$  of  $\varphi$  with  $\varphi_0 = \varphi$ ,

$$\left. \frac{d}{dt} \right|_{t=0} E_b(\varphi_t) = - \int_M h(\tau_b(\varphi), V) d\theta \wedge (d\theta)^n = 0, \quad (2.9)$$

where  $V \in \Gamma(\varphi^{-1}TN)$  is defined by  $V(x) = \left. \frac{d}{dt} \right|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N$ , ( $x \in M$ ). Here,  $\tau_b(\varphi)$  is the *pseudo tension field* which is given by

$$\tau_b(\varphi) = \sum_{i=1}^{2n} B_\varphi(X_i, X_i), \quad (2.10)$$

where  $B_\varphi(X, Y)$  ( $X, Y \in \mathfrak{X}(M)$ ) is the second fundamental form (2.2) for a  $C^\infty$  map of  $(M, g_\theta)$  into  $(N, h)$ . Then,  $\varphi$  is *pseudo harmonic* if  $\tau_b(\varphi) = 0$ .

The second variational formula of  $E_b$  is given as follows ([5], p.733):

$$\left. \frac{d^2}{dt^2} \right|_{t=0} E_b(\varphi_t) = \int_M h(J_b(V), V) \theta \wedge (d\theta)^n, \quad (2.11)$$

where  $J_b$  is a subelliptic operator acting on  $\Gamma(\varphi^{-1}TN)$  given by

$$J_b(V) = \Delta_b V - \mathcal{R}_b(V). \quad (2.12)$$

Here, for  $V \in \Gamma(\varphi^{-1}TN)$ ,

$$\begin{cases} \Delta_b V = (\bar{\nabla}^H)^* \bar{\nabla}^H V = - \sum_{i=1}^{2n} \{ \bar{\nabla}_{X_i} (\bar{\nabla}_{X_i} V) - \bar{\nabla}_{\nabla_{X_i} X_i} V \}, \\ \mathcal{R}_b(V) = \sum_{i=1}^{2n} R^h(V, d\varphi(X_i)) d\varphi(X_i), \end{cases} \quad (2.13)$$

where  $\nabla$  is the Tanaka-Webster connection, and  $\bar{\nabla}$ , the induced connection on  $\phi^{-1}TN$  induced from the Levi-Civita connection  $\nabla^h$ , and  $\{X_i\}_{i=1}^{2n}$ , a local orthonormal frame field on  $(H(M), g_\theta)$ , respectively. Here,  $(\bar{\nabla}^H)_X V := \bar{\nabla}_{X^H} V$  ( $X \in \mathfrak{X}(M)$ ,  $V \in \Gamma(\phi^{-1}TN)$ ), corresponding to the decomposition  $X = X^H + g_\theta(X, T) T$  ( $X^H \in H(M)$ ), and define  $\pi_H(X) = X^H$  ( $X \in T_x(M)$ ), and  $(\bar{\nabla}^H)^*$  is the formal adjoint of  $\bar{\nabla}^H$ .

Dragomir and Montaldo [10] introduced the *pseudo bienergy* given by

$$E_{b,2}(\varphi) = \frac{1}{2} \int_M h(\tau_b(\varphi), \tau_b(\varphi)) \theta \wedge (d\theta)^n, \quad (2.14)$$

where  $\tau_b(\varphi)$  is the *pseudo tension field* of  $\varphi$ . They gave the first variational formula of  $E_{b,2}$  as follows ([10], p.227):

$$\left. \frac{d}{dt} \right|_{t=0} E_{b,2}(\varphi_t) = - \int_M h(\tau_{b,2}(\varphi), V) \theta \wedge (d\theta)^n, \quad (2.15)$$

where  $\tau_{b,2}(\varphi)$  is called the *pseudo bitension field* given by

$$\tau_{b,2}(\varphi) = \Delta_b(\tau_b(\varphi)) - \sum_{i=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_i)) d\varphi(X_i). \quad (2.16)$$

Then, a smooth map  $\varphi$  of  $(M, g_\theta)$  into  $(N, h)$  is said to be *pseudo biharmonic* if  $\tau_{b,2}(\varphi) = 0$ . By definition, a pseudo harmonic map is always pseudo biharmonic.

### 3. GENERALIZED CHEN'S CONJECTURE FOR PSEUDO BIHARMONIC MAPS

**3.1** First, let us recall the usual Weitzenbeck formula for a  $C^\infty$  map from a Riemannian manifold  $(M, g)$  of  $(2n + 1)$  dimension into a Riemannian manifold  $(N, h)$ :

**Lemma 3.1.** *(The Weitzenbeck formula) For every  $C^\infty$  map  $\varphi$  of  $(M, g)$  of  $(2n + 1)$ -dimension into a Riemannian manifold  $(N, h)$ , the Hodge Laplacian  $\Delta$  acting on the 1-form  $d\varphi$ , regarded as a  $\varphi^{-1}TN$ -valued 1 form,  $d\varphi \in \Gamma(T^*M \otimes \varphi^{-1}TN)$ , we have*

$$\Delta d\varphi = \widetilde{\nabla}^* \widetilde{\nabla} d\varphi + S. \quad (3.1)$$

Here, let us recall the rough Laplacian

$$\widetilde{\nabla}^* \widetilde{\nabla} := \sum_{k=1}^{2n+1} \left\{ \widetilde{\nabla}_{e_k} \widetilde{\nabla}_{e_k} - \widetilde{\nabla}_{\nabla^g_{e_k} e_k} \right\} \quad (3.2)$$

$$S(X) := -(\widetilde{R}(X, e_k)d\varphi)(e_k), \quad (X \in \mathfrak{X}(M)). \quad (3.3)$$

Here,  $\nabla^g, \nabla^h$  are the Levi-Civita connections of  $(M, g)$ ,  $(N, h)$ , and  $\widetilde{\nabla}$  is the induced connection on  $T^*M \otimes \varphi^{-1}TN$  defined by  $(\widetilde{\nabla}_X d\varphi)(Y) = \widetilde{\nabla}_X d\varphi(Y) - d\varphi(\nabla^g_X Y)$ ,  $\widetilde{\nabla}$  is the induced connection on  $\varphi^{-1}TN$  given by  $\widetilde{\nabla}_X d\varphi(Y) = \nabla^h_{d\varphi(X)} d\varphi(Y)$ ,  $(X, Y \in \mathfrak{X}(M))$ , and  $\{e_k\}_{k=1}^{2n+1}$  is a locally defined orthonormal vector field on  $(M, g)$ . The curvature tensor field  $\widetilde{R}$  in (3.3) is defined by

$$\begin{aligned} (\widetilde{R}(X, Y)d\varphi)(Z) &:= \overline{R}(X, Y) d\varphi(Z) - d\varphi(R^g(X, Y)Z) \\ &= R^h(d\varphi(X), d\varphi(Y))d\varphi(Z) - d\varphi(R^g(X, Y)Z), \end{aligned}$$

for  $X, Y, Z \in \mathfrak{X}(M)$ , where  $\overline{R}$ ,  $R^g$ , and  $R^h$  are the curvature tensors of the induced connection  $\widetilde{\nabla}$ ,  $\nabla^g$  and  $\nabla^h$ , respectively.

Notice that for an isometric immersion  $\varphi : (M, g) \rightarrow (N, h)$ , it holds that

$$(\widetilde{\nabla}_X d\varphi)(Y) = B_\varphi(X, Y), \quad (X, Y \in \mathfrak{X}(M)). \quad (3.4)$$

**3.2** In this part, we first raise the CR analogue of the generalized Chen's conjecture, and settle it for pseudo biharmonic maps with finite pseudo energy and finite pseudo bienergy.

Let us recall a strictly pseudoconvex CR manifold (possibly non compact)  $(M, g_\theta)$  of  $(2n + 1)$ -dimension, and the Webster Riemannian metric  $g_\theta$  given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1$$

for  $X, Y \in H(M)$ . Recall the material on the Levi-Civita connection  $\nabla^{g_\theta}$  of  $(M, g_\theta)$ . Due to Lemma 1.3, Page 38 in [11], it holds that,

$$\nabla^{g_\theta} = \nabla + (\Omega - A) \otimes T + \tau \otimes \theta + 2\theta \odot J, \quad (3.5)$$

where  $\nabla$  is the Tanaka-Webster connection,  $\Omega = d\theta$ , and  $A(X, Y) = g_\theta(\tau X, Y)$ ,  $\tau X = T_\nabla(T, X)$ , and  $T_\nabla$  is the torsion tensor of  $\nabla$ . And also,  $(\tau \otimes \theta)(X, Y) = \theta(Y) \tau X$ ,  $(\theta \odot J)(X, Y) = \frac{1}{2} \{\theta(X) JY + \theta(Y) JX\}$  for all vector fields  $X, Y$  on  $M$ . Here,  $J$  is the complex structure on  $H(M)$  and is extended as an endomorphism on  $(M)$  by  $JT = 0$ .

Then, we have

$$\nabla_{X_k}^{g_\theta} X_k = \nabla_{X_k} X_k - A(X_k, X_k) T, \quad (3.6)$$

$$\nabla_T^{g_\theta} T = 0, \quad (3.7)$$

where  $\{X_k\}_{k=1}^{2n}$  is a locally defined orthonormal frame field on  $H(M)$  with respect to  $g_\theta$ , and  $T$  is the characteristic vector field of  $(M, g_\theta)$ . For (3.6), it follows from that  $\Omega(X_k, X_k) = 0$ ,  $(\tau \otimes \theta)(X_k, X_k) = 0$ , and  $(\theta \odot J)(X_k, X_k) = 0$  since  $\theta(X_k) = 0$ . For (3.7), notice that the Tanaka-Webster connection  $\nabla$  satisfies  $\nabla_T T = 0$ , and also  $\tau T = 0$  and  $JT = 0$ , so that  $\Omega(T, T) = 0$ ,  $A(T, T) = 0$ ,  $(\tau \otimes \theta)(T, T) = 0$   $(\theta \odot J)(T, T) = 0$  which imply (3.7).

For (3.2) in the Weitenbeck formula in Lemma 3.1, by taking  $\{X_k$  ( $k = 1, \dots, 2n$ ),  $T\}$ , as an orthonormal basis  $\{e_k\}$  of our  $(M, g^\theta)$ , and due to (3.6) and (3.7), we have

$$\begin{aligned} (\widetilde{\nabla}^* \widetilde{\nabla} d\varphi)(X) &= (\widetilde{\Delta}_b d\varphi)(X) \\ &= - \sum_{k=1}^{2n+1} \{ \widetilde{\nabla}_{e_k} \widetilde{\nabla}_{e_k} - \widetilde{\nabla}_{\nabla_{e_k}^{g_\theta} e_k} \} d\varphi(X) \\ &= - \sum_{k=1}^{2n} \{ \widetilde{\nabla}_{X_k} \widetilde{\nabla}_{X_k} - \widetilde{\nabla}_{\nabla_{X_k}^{g_\theta} X_k} \} d\varphi(X) \\ &\quad - \{ \widetilde{\nabla}_T \widetilde{\nabla}_T - \widetilde{\nabla}_{\nabla_T^{g_\theta} T} \} d\varphi(X) \\ &= - \sum_{k=1}^{2n} \{ \widetilde{\nabla}_{X_k} \widetilde{\nabla}_{X_k} - \widetilde{\nabla}_{\nabla_{X_k} X_k} \} d\varphi(X) \\ &\quad - \{ \widetilde{\nabla}_T \widetilde{\nabla}_T + \sum_{k=1}^{2n} A(X_k, X_k) \widetilde{\nabla}_T \} d\varphi(X) \\ &= - \sum_{k=1}^{2n} \{ \widetilde{\nabla}_{X_k} \widetilde{\nabla}_{X_k} - \widetilde{\nabla}_{\nabla_{X_k} X_k} \} d\varphi(X) - \widetilde{\nabla}_T \widetilde{\nabla}_T d\varphi(X). \end{aligned} \quad (3.8)$$

since  $\sum_{k=1}^{2n} A(X_k, X_k) = 0$  (cf. [11], p. 35).



For (3.3) in the Weitzenbeck formula in Lemma 3.1, we have

$$\begin{aligned}
S(X) &= - \sum_{k=1}^{2n+1} (\tilde{R}(X, e_k) d\varphi)(e_k) \\
&= - \sum_{k=1}^{2n} (\tilde{R}(X, X_k) d\varphi)(X_k) - (\tilde{R}(X, T) d\varphi)(T) \\
&= - \sum_{k=1}^{2n} \left\{ R^h(d\varphi(X), d\varphi(X_k)) d\varphi(X_k) - d\varphi(R^{g_\theta}(X, X_k) X_k) \right\} \\
&\quad - \left\{ R^h(d\varphi(X), d\varphi(T)) d\varphi(T) - d\varphi(R^{g_\theta}(X, T) T) \right\}. \tag{3.9}
\end{aligned}$$

And, we have the following formulas for (3.1) in our case,

$$\begin{aligned}
\Delta d\varphi(X) &= d d^* d\varphi(X) \\
&= -d \tau(\varphi)(X) \\
&= -\bar{\nabla}_X \tau(\varphi). \tag{3.10}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-(\tilde{\Delta}_b d\varphi)(X) &= \sum_{k=1}^{2n} \left\{ \tilde{\nabla}_{X_k} \tilde{\nabla}_{X_k} - \tilde{\nabla}_{\nabla_{X_k} X_k} \right\} d\varphi(X) \\
&= -(\Delta d\varphi)(X) + S(X) - \tilde{\nabla}_T \tilde{\nabla}_T d\varphi(X) \\
&= \bar{\nabla}_X \tau(\varphi) \\
&\quad - \sum_{k=1}^{2n} \left\{ R^h(d\varphi(X), d\varphi(X_k)) d\varphi(X_k) - d\varphi(R^{g_\theta}(X, X_k) X_k) \right\} \\
&\quad - \left\{ R^h(d\varphi(X), d\varphi(T)) d\varphi(T) - d\varphi(R^{g_\theta}(X, T) T) \right\} \\
&\quad - \tilde{\nabla}_T \tilde{\nabla}_T d\varphi(X). \tag{3.11}
\end{aligned}$$

**3.3** Let us consider the generalized B.-Y. Chen's conjecture for pseudo biharmonic maps which is CR analogue of the usual generalized Chen's conjecture for biharmonic maps:

**The CR analogue of the generalized B.-Y. Chen's conjecture for pseudo biharmonic maps:**

*Let  $(M, g_\theta)$  be a complete strictly pseudoconvex CR manifold, and assume that  $(N, h)$  is a Riemannian manifold of non-positive curvature.*

*Then, every pseudo biharmonic isometric immersion  $\varphi : (M, g_\theta) \rightarrow (N, h)$  must be pseudo harmonic.*

In this section, we want to show that the above conjecture is true under the finiteness of the pseudo energy and pseudo bienergy.

**Theorem 3.2.** *Assume that  $\varphi$  is a pseudo biharmonic map of a strictly pseudoconvex complete CR manifold  $(M, g_\theta)$  into another Riemannian manifold  $(N, h)$  of non positive curvature.*

*If  $\varphi$  has finite pseudo bienergy  $E_{b,2}(\varphi) < \infty$  and finite pseudo energy  $E_b(\varphi) < \infty$ , then it is pseudo harmonic, i.e.,  $\tau_b(\varphi) = 0$ .*

(Proof of Theorem 3.2) The proof is divided into several steps.

(The first step) For an arbitrarily fixed point  $x_0 \in M$ , let  $B_r(x_0) = \{x \in M : r(x) < r\}$  where  $r(x)$  is a distance function on  $(M, g_\theta)$ , and let us take a cut off function  $\eta$  on  $(M, g_\theta)$ , i.e.,

$$\begin{cases} 0 \leq \eta(x) \leq 1 & (x \in M), \\ \eta(x) = 1 & (x \in B_r(x_0)), \\ \eta(x) = 0 & (x \notin B_{2r}(x_0)), \\ |\nabla^{g_\theta} \eta| \leq \frac{2}{r} & (x \in M), \end{cases} \quad (3.12)$$

where  $r, \nabla^{g_\theta}$  are the distance function, the Levi-Civita connection of  $(M, g_\theta)$ , respectively. Assume that  $\varphi : (M, g_\theta) \rightarrow (N, h)$  is a pseudo biharmonic map, i.e.,

$$\begin{aligned} \tau_{b,2}(\varphi) &= J_b(\tau_b(\varphi)) \\ &= \Delta_b(\tau_b(\varphi)) - \sum_{j=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_j)) d\varphi(X_j) \\ &= 0. \end{aligned} \quad (3.13)$$

(The second step) Then, we have

$$\begin{aligned} &\int_M \langle \Delta_b(\tau_b(\varphi)), \eta^2 \tau_b(\varphi) \rangle \theta \wedge (d\theta)^n \\ &= \int_M \eta^2 \sum_{j=1}^{2n} \langle R^h(\tau_b(\varphi), d\varphi(X_j)) d\varphi(X_j), \tau_b(\varphi) \rangle \theta \wedge (d\theta)^n \\ &\leq 0 \end{aligned} \quad (3.14)$$

since  $(N, h)$  has the non-positive sectional curvature. But, for the left hand side of (3.14), it holds that

$$\begin{aligned}
& \int_M \langle \Delta_b(\tau_b(\varphi)), \eta^2 \tau_b(\varphi) \rangle \theta \wedge (d\theta)^n \\
&= \int_M \langle \bar{\nabla}^H \tau_b(\varphi), \bar{\nabla}^H (\eta^2 \tau_b(\varphi)) \rangle \theta \wedge (d\theta)^n \\
&= \int_M \sum_{j=1}^{2n} \langle \bar{\nabla}_{X_j} \tau_b(\varphi), \bar{\nabla}_{X_j} (\eta^2 \tau_b(\varphi)) \rangle \theta \wedge (d\theta)^n.
\end{aligned} \tag{3.15}$$

Here, let us recall, for  $V, W \in \Gamma(\varphi^{-1}TN)$ ,

$$\langle \bar{\nabla}^H V, \bar{\nabla}^H W \rangle = \sum_{\alpha} \langle \bar{\nabla}_{e_{\alpha}}^H V, \bar{\nabla}_{e_{\alpha}}^H W \rangle = \sum_{j=1}^{2n} \langle \bar{\nabla}_{X_j} V, \bar{\nabla}_{X_j} W \rangle,$$

where  $\{e_{\alpha}\}$  is a locally defined orthonormal frame field of  $(M, g_{\theta})$  and  $\bar{\nabla}_X^H W$  ( $X \in \mathfrak{X}(M)$ ,  $W \in \Gamma(\varphi^{-1}TN)$ ) is defined by

$$\bar{\nabla}_X^H W = \sum_j \{ (X^H f_j) V_j + f_j \bar{\nabla}_{X^H} V_j \}$$

for  $W = \sum_j f_j V_j$  ( $f_j \in C^{\infty}(M)$  and  $V_j \in \Gamma(\varphi^{-1}TN)$ ). Here,  $X^H$  is the  $H(M)$ -component of  $X$  corresponding to the decomposition of  $T_x(M) = H_x(M) \oplus \mathbb{R}T_x$  ( $x \in M$ ), and  $\bar{\nabla}$  is the induced connection of  $\varphi^{-1}TN$  from the Levi-Civita connection  $\nabla^h$  of  $(N, h)$ .

Since

$$\bar{\nabla}_{X_j}(\eta^2 \tau_b(\varphi)) = 2\eta X_j \eta \tau_b(\varphi) + \eta^2 \bar{\nabla}_{X_j} \tau_b(\varphi), \tag{3.16}$$

the right hand side of (3.15) is equal to

$$\begin{aligned}
& \int_M \eta^2 \sum_{j=1}^{2n} \left| \bar{\nabla}_{X_j} \tau_b(\varphi) \right|^2 \theta \wedge (d\theta)^n \\
&+ 2 \int_M \sum_{j=1}^{2n} \langle \eta \bar{\nabla}_{X_j} \tau_b(\varphi), (X_j \eta) \tau_b(\varphi) \rangle \theta \wedge (d\theta)^n.
\end{aligned} \tag{3.17}$$

Therefore, together with (3.14), we have

$$\begin{aligned}
& \int_M \eta^2 \sum_{j=1}^{2n} \left| \bar{\nabla}_{X_j} \tau_b(\varphi) \right|^2 \theta \wedge (d\theta)^n \\
&\leq -2 \int_M \sum_{j=1}^{2n} \langle \eta \bar{\nabla}_{X_j} \tau_b(\varphi), (X_j \eta) \tau_b(\varphi) \rangle \theta \wedge (d\theta)^n \\
&=: -2 \int_M \sum_{j=1}^{2n} \langle V_j, W_j \rangle \theta \wedge (d\theta)^n,
\end{aligned} \tag{3.18}$$

where we define  $V_j, W_j \in \Gamma(\varphi^{-1}TN)$  ( $j = 1, \dots, 2n$ ) by

$$V_j := \eta \bar{\nabla}_{X_j} \tau_b(\varphi), \quad W_j := (X_j \eta) \tau_b(\varphi).$$

Then, since it holds that  $0 \leq \left| \sqrt{\epsilon} V_i \pm \frac{1}{\sqrt{\epsilon}} W_i \right|^2$  for every  $\epsilon > 0$ , we have,

the right hand side of (3.18)

$$\leq \epsilon \int_M \sum_{j=1}^{2n} |V_j|^2 \theta \wedge (d\theta)^n + \frac{1}{\epsilon} \int_M \sum_{j=1}^{2n} |W_j|^2 \theta \wedge (d\theta)^n \quad (3.19)$$

for every  $\epsilon > 0$ . By taking  $\epsilon = \frac{1}{2}$ , we obtain

$$\begin{aligned} & \int_M \eta^2 \sum_{j=1}^{2n} |\bar{\nabla}_{X_j} \tau_b(\varphi)|^2 \theta \wedge (d\theta)^n \\ & \leq \frac{1}{2} \int_M \sum_{j=1}^{2n} \eta^2 |\bar{\nabla}_{X_j} \tau_b(\varphi)|^2 \theta \wedge (d\theta)^n + 2 \int_M \sum_{j=1}^{2n} |X_j \eta|^2 |\tau_b(\varphi)|^2 \theta \wedge (d\theta)^n. \end{aligned} \quad (3.20)$$

Therefore, we obtain, due to the properties that  $\eta = 1$  on  $B_r(x_0)$ , and  $\sum_{j=1}^{2n} |X_j \eta|^2 \leq |\nabla^{g_\theta} \eta|^2 \leq \left(\frac{2}{r}\right)^2$ ,

$$\begin{aligned} \int_{B_r(x_0)} \sum_{j=1}^{2n} |\bar{\nabla} \tau_b(\varphi)|^2 \theta \wedge (d\theta)^n & \leq \int_M \eta^2 \sum_{j=1}^{2n} |\bar{\nabla}_{X_j} \tau_b(\varphi)|^2 \theta \wedge (d\theta)^n \\ & \leq 4 \int_M \sum_{j=1}^{2n} |X_j \eta|^2 |\tau_b(\varphi)|^2 \theta \wedge (d\theta)^n \\ & \leq \frac{16}{r^2} \int_M |\tau_b(\varphi)|^2 \theta \wedge (d\theta)^n. \end{aligned} \quad (3.21)$$

(The third step) By our assumption that  $E_{b,2}(\varphi) = \frac{1}{2} \int_M |\tau_b(\varphi)|^2 \theta \wedge (d\theta)^n < \infty$  and  $(M, g_\theta)$  is complete, if we let  $r \rightarrow \infty$ , then  $B_r(x_0)$  goes to  $M$ , and the right hand side of (3.21) goes to zero. We have

$$\int_M \sum_{j=1}^{2n} |\bar{\nabla}_{X_j} \tau_b(\varphi)|^2 \theta \wedge (d\theta)^n = 0. \quad (3.22)$$

This implies that

$$\bar{\nabla}_X \tau_b(\varphi) = 0 \quad (\text{for all } X \in H(M)). \quad (3.23)$$

(The forth step) Let us take a 1 form  $\alpha$  on  $M$  defined by

$$\alpha(X) = \begin{cases} \langle d\varphi(X), \tau_b(\varphi) \rangle, & (X \in H(M)), \\ 0 & (X = T). \end{cases}$$

Then, we have

$$\begin{aligned} \int_M |\alpha| \theta \wedge (d\theta)^n &= \int_M \left( \sum_{j=1}^{2n} |\alpha(X_j)|^2 \right)^{\frac{1}{2}} \theta \wedge (d\theta)^n \\ &\leq \left( |d_b\varphi|^2 \theta \wedge (d\theta)^n \right)^{\frac{1}{2}} \left( \int_M |\tau_b(\varphi)|^2 \theta \wedge (d\theta)^n \right)^{\frac{1}{2}} \\ &= 2 \sqrt{E_b(\varphi) E_{b,2}(\varphi)} < \infty, \end{aligned} \quad (3.24)$$

where we put  $d_b\varphi := \sum_{i=1}^{2n} d\varphi(X_i) \otimes X_i$ ,

$$|d_b\varphi|^2 = \sum_{i,j=1}^{2n} g_\theta(X_i, X_j) h(d\varphi(X_i), d\varphi(X_j)) = \sum_{i=1}^{2n} h(d\varphi(X_i), d\varphi(X_i)),$$

and

$$E_b(\varphi) = \frac{1}{2} \int_M |d_b\varphi|^2 \theta \wedge (d\theta)^n. \quad (3.25)$$

Furthermore, let us define a  $C^\infty$  function  $\delta_b\alpha$  on  $M$  by

$$\delta_b\alpha = - \sum_{j=1}^{2n} (\nabla_{X_j}\alpha)(X_j) = - \sum_{j=1}^{2n} \left\{ X_j(\alpha(X_j)) - \alpha(\nabla_{X_j}X_j) \right\}, \quad (3.26)$$

where  $\nabla$  is the Tanaka-Webster connection. Notice that

$$\begin{aligned} \operatorname{div}(\alpha) &= \sum_{j=1}^{2n} (\nabla_{X_j}^{g_\theta}\alpha)(X_j) + (\nabla_T^{g_\theta}\alpha)(T) \\ &= \sum_{j=1}^{2n} \left\{ X_j(\alpha \circ \pi_H(X_j)) - \alpha \circ \pi_H(\nabla_{X_j}^{g_\theta}X_j) \right\} \\ &\quad + T(\alpha \circ \pi_H(T)) - \alpha \circ \pi_H(\nabla_T^{g_\theta}T) \\ &= \sum_{j=1}^{2n} \left\{ X_j(\alpha(X_j)) - \alpha(\pi_H(\nabla_{X_j}^{g_\theta}X_j)) \right\} \\ &= \sum_{j=1}^{2n} \left\{ X_j(\alpha(X_j)) - \alpha(\nabla_{X_j}X_j) \right\} \\ &= -\delta_b\alpha, \end{aligned} \quad (3.27)$$

where  $\pi_H : T_x(M) \rightarrow H_x(M)$  is the natural projection. We used the facts that  $\nabla_T^{g_\theta}T = 0$ , and  $\pi_H(\nabla_X^{g_\theta}Y) = \nabla_X Y$  ( $X, Y \in H(M)$ ) ([2],

p.37). Here, recall again  $\nabla^{g_\theta}$  is the Levi-Civita connection of  $g_\theta$ , and  $\nabla$  is the Tanaka-Webster connection. Then, we have, for (3.26),

$$\begin{aligned}
\delta_b \alpha &= - \sum_{j=1}^{2n} \left\{ X_j \langle d\varphi(X_j), \tau_b(\varphi) \rangle - \langle d\varphi(\nabla_{X_j} X_j), \tau_b(\varphi) \rangle \right\} \\
&= - \sum_{j=1}^{2n} \left\{ \langle \bar{\nabla}_{X_j}(d\varphi(X_j)), \tau_b(\varphi) \rangle + \langle d\varphi(X_j), \bar{\nabla}_{X_j} \tau_b(\varphi) \rangle \right. \\
&\quad \left. - \langle d\varphi(\nabla_{X_j} X_j), \tau_b(\varphi) \rangle \right\} \\
&= - \left\langle \sum_{j=1}^{2n} \left\{ \bar{\nabla}_{X_j}(d\varphi(X_j)) - d\varphi(\nabla_{X_j} X_j) \right\}, \tau_b(\varphi) \right\rangle \\
&= - |\tau_b(\varphi)|^2.
\end{aligned} \tag{3.28}$$

We used (3.23)  $\bar{\nabla}_{X_j} \tau_b(\varphi) = 0$  to derive the last second equality of (3.28). Then, due to (3.28), we have for  $E_{b,2}(\varphi)$ ,

$$\begin{aligned}
E_{b,2}(\varphi) &= \frac{1}{2} \int_M |\tau_b(\varphi)|^2 \theta \wedge (d\theta)^n \\
&= -\frac{1}{2} \int_M \delta_b \alpha \theta \wedge (d\theta)^n \\
&= \frac{1}{2} \int_M \operatorname{div}(\alpha) \theta \wedge (d\theta)^n \\
&= 0.
\end{aligned} \tag{3.29}$$

In the last equality, we used Gaffney's theorem ([28], p. 271, [14]).

Therefore, we obtain  $\tau_b(\varphi) \equiv 0$ , i.e.,  $\varphi$  is pseudo harmonic.  $\square$

#### 4. PARALLEL PSEUDO BIHARMONIC ISOMETRIC IMMERSION INTO RANK ONE SYMMETRIC SPACES

On the contrary of the Section Three, we consider isometric immersions into the unit sphere or the complex projective spaces which are pseudo biharmonic. One of the main theorem of this section is as follows:

**Theorem 4.1.** *Let  $\varphi : (M, g_\theta) \rightarrow S^{2n+2}(1)$  be an isometric immersion of a CR manifold  $(M, g_\theta)$  of  $(2n+1)$ -dimension into the unit sphere  $S^{2n+2}(1)$  of constant sectional curvature 1 and  $(2n+2)$ -dimension. Assume that  $\varphi$  admits a parallel pseudo mean curvature vector field with non-zero pseudo mean curvature. The following equivalences hold: The immersion  $\varphi$  is pseudo biharmonic if and only if*

$$\sum_{i=1}^{2n} \lambda_i^2 = 2n \tag{4.1}$$

if and only if

$$\|B_\varphi|_{H(M) \times H(M)}\|^2 = 2n, \quad (4.2)$$

where  $\lambda_i$  ( $1 \leq i \leq 2n+1$ ) are the principal curvatures of the immersion  $\varphi$  whose  $\lambda_{2n+1}$  corresponds to the characteristic vector field  $T$  of  $(M, g_\theta)$ , and  $B_\varphi|_{H(M) \times H(M)}$  is the restriction of the second fundamental form of  $\varphi$  to the orthogonal complement  $H(M)$  of  $T$  in the tangent space  $(T_x(M), g_\theta)$ .

As applications of this theorem, we will give pseudo biharmonic immersions into the unit sphere which are not pseudo harmonic.

The other case of rank one symmetric space is the complex projective space  $\mathbb{P}^{n+1}(c)$ . We obtain the following theorem:

**Theorem 4.2.** *Let  $\varphi : (M^{2n+1}, g_\theta) \rightarrow \mathbb{P}^{n+1}(c)$  be an isometric immersion of CR manifold  $(M, g_\theta)$  into the complex projective space  $\mathbb{P}^{n+1}(c)$  of constant holomorphic sectional curvature  $c$  and complex  $(n+1)$ -dimension. Assume that  $\varphi$  has parallel pseudo-mean curvature vector field with non-zero pseudo mean curvature. Then, the following equivalence relation holds: The immersion  $\varphi$  is pseudo-biharmonic if and only if the following hold:*

*Either (1)  $J(d\varphi(T))$  is tangent to  $\varphi(M)$  and*

$$\|B_\varphi|_{H(M) \times H(M)}\|^2 = \frac{c}{4}(2n+3), \quad (4.3)$$

*or (2)  $J(d\varphi(T))$  is normal to  $\varphi(M)$  and*

$$\|B_\varphi|_{H(M) \times H(M)}\|^2 = \frac{c}{4}(2n) = c \frac{n}{2}. \quad (4.4)$$

As applications of this theorem, we will give pseudo biharmonic, but not pseudo harmonic immersions  $(M, g_\theta)$  into the complex projective space  $\mathbb{P}^{n+1}(c)$ .

## 5. ADMISSIBLE IMMERSIONS OF STRONGLY PSEUDOCONVEX CR MANIFOLDS

In this section, we introduce the notion of admissible isometric immersion of strongly pseudoconvex CR manifold  $(M, g_\theta)$ , and will show the following two lemmas related to  $\Delta_b(\tau_b(\varphi))$  which are necessary to prove main theorems.

**Definition 5.1.** Let  $(M^{2n+1}, g_\theta)$  be a strictly pseudoconvex CR manifold, and  $T_x M = H_x(M) \oplus \mathbb{R}T_x$ ,  $(x \in M)$ , the orthogonal decomposition of the tangent space  $T_x M$  ( $x \in M$ ), where  $T$  is the characteristic vector field of  $(M^{2n+1}, g_\theta)$ ,  $\varphi : (M^{2n+1}, g_\theta) \rightarrow (N, h)$  be an isometric immersion. The immersion  $\varphi$  is called to be admissible if the second fundamental form  $B_\varphi$  satisfies that

$$B_\varphi(X, T) = 0 \quad (5.1)$$

for all vector field  $X$  in  $H(M)$ .

The following clarifies the meaning of the admissibility condition:

**Proposition 5.2.** Let  $\varphi$  be an isometric immersion of a strongly pseudoconvex CR manifold  $(M^{2n+1}, g_\theta)$  into another Riemannian manifold  $(N, h)$ . Then,  $\varphi$  is admissible if and only if

(1)  $d\varphi(T_x)$  ( $x \in M$ ) is a principal curvature vector field along  $\varphi$  with some principal curvature  $\lambda(x)$  ( $x \in M$ ).

This is equivalent the following:

(2) The shape operator  $A_\xi$  of the immersion  $\varphi : (M, g_\theta) \rightarrow (N, h)$  preserves  $H_x(M)$  ( $x \in M$ ) invariantly for a normal vector field  $\xi$ .

(Proof of Proposition 5.2) We first note for every normal vector field  $\xi$  of the isometric immersion  $\varphi : (M, g_\theta) \rightarrow (N, h)$ , it holds that

$$\langle B_\varphi(X, T), \xi \rangle = g_\theta(A_\xi X, T) = g_\theta(X, A_\xi T), \quad (X \in H_x(M)). \quad (\#)$$

Thus, if  $\varphi$  is admissible, then the left hand side of  $(\#)$  vanishes, then we have immediately that

$$\begin{cases} A_\xi X \in H_x(M) & (X \in H_x(M)), \\ A_\xi T_x = \lambda(x) T_x & (\text{for some real number } \lambda(x)). \end{cases} \quad (b)$$

Conversely, if one of the conditions of  $(b)$  holds, then it turns out immediately that  $\varphi$  is admissible.  $\square$

The following two lemmas will be essential to us later.

**Lemma 5.3.** Let  $\varphi : (M^{2n+1}, g_\theta) \rightarrow (N, h)$  be an admissible isometric immersion with parallel pseudo mean curvature vector field. Then, the pseudo tension field  $\tau_b(\varphi)$  satisfies that

$$\begin{aligned} -\Delta_b(\tau_b(\varphi)) &= \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_i) \rangle d\varphi(X_i) \\ &\quad + \langle \widetilde{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle (\widetilde{\nabla}_{X_i} d\varphi)(X_j), \end{aligned} \quad (5.2)$$

where  $\{X_j\}_{j=1}^{2n}$  is a local orthonormal frame field of  $H(M)$  with respect to  $g_\theta$ .



**Lemma 5.4.** *Under the same assumptions of the above lemma, we have*

$$\begin{aligned} -\Delta_b(\tau_b(\varphi)) &= \left\langle \tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(X_k))d\varphi(X_k) \right\rangle d\varphi(X_j) \\ &\quad + \left\langle \tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(T))d\varphi(T) \right\rangle d\varphi(X_j) \\ &\quad - \left\langle \tau_b(\varphi), \left( \widetilde{\nabla}_{X_i} d\varphi \right) (X_j) \right\rangle \left( \widetilde{\nabla}_{X_i} d\varphi \right) (X_j), \end{aligned} \quad (5.3)$$

where  $R^h(U, V)W$  is the curvature tensor field of  $(N, h)$  defined by  $R^h(U, V)W = \nabla_U^h(\nabla_V^h W) - \nabla_V^h(\nabla_U^h W) - \nabla_{[U, V]}^h W$  for vector fields  $U, V, W$  on  $N$ , and  $\nabla^h$  is the Levi-Civita connection of  $(N, h)$ .

(Proof of Lemma 5.3) The proof is divided into several steps.

(The first step) Since we assume the pseudo mean curvature vector field  $\tau_b(\varphi)$  is parallel, i.e.,  $\widetilde{\nabla}_X^\perp \tau_b(\varphi) = 0$  ( $X \in \mathfrak{X}(M)$ ), the induced connection  $\overline{\nabla}$  of the Levi-Civita connection  $\nabla^h$  to the induced bundle  $\varphi^{-1}TN$  satisfies that, for all  $X \in \mathfrak{X}(M)$ ,

$$\overline{\nabla}_X \tau_b(\varphi) = \overline{\nabla}_X^\top \tau_b(\varphi) + \overline{\nabla}_X^\perp \tau_b(\varphi) = \overline{\nabla}_X^\top \tau_b(\varphi) \in \Gamma(\varphi_* TM).$$

Then, we have, for all  $X \in H(M)$ ,

$$\begin{aligned} \overline{\nabla}_X \tau_b(\varphi) &= \sum_{j=1}^{2n} \langle \overline{\nabla}_X \tau_b(\varphi), d\varphi(X_j) \rangle d\varphi(X_j) + \langle \overline{\nabla}_X \tau_b(\varphi), d\varphi(T) \rangle d\varphi(T) \\ &= \sum_{j=1}^{2n} \langle \overline{\nabla}_X \tau_b(\varphi), d\varphi(X_j) \rangle d\varphi(X_j). \end{aligned} \quad (5.4)$$

Due to the assumption of the admissibility of  $\varphi$ , for all  $X \in H(M)$ ,

$$\langle \overline{\nabla}_X \tau_b(\varphi), d\varphi(T) \rangle = X \langle \tau_b(\varphi), d\varphi(T) \rangle - \langle \tau_b(\varphi), \overline{\nabla}_X d\varphi(T) \rangle = 0. \quad (5.5)$$

In fact,  $\tau_b(\varphi) = \sum_{i=1}^{2n} B_\varphi(X_i, X_i)$  is orthogonal to  $d\varphi(TM)$  with respect to  $\langle \cdot, \cdot \rangle$ , we have  $\langle \tau_b(\varphi), d\varphi(T) \rangle = 0$ . So, the first term of (5.5) vanishes. By the admissibility of  $\varphi$ , for all  $X \in H(M)$ ,

$$0 = B_\varphi(X, T) = \overline{\nabla}_X d\varphi(T) - d\varphi(\nabla_X^{g_\varphi} T), \quad (5.6)$$

so that  $\overline{\nabla}_X d\varphi(T)$  is tangential, which implies that

$$\langle \tau_b(\varphi), \overline{\nabla}_X d\varphi(T) \rangle = 0.$$

We have (5.5), and then (5.4).

(The second step) We calculate  $-\Delta_b(\tau_b(\varphi))$ . We have by (5.4),

$$\begin{aligned}
-\Delta_b(\tau_b(\varphi)) &= \sum_{i=1}^{2n} \left\{ \bar{\nabla}_{X_i}(\bar{\nabla}_{X_i}\tau_b(\varphi)) - \bar{\nabla}_{\nabla_{X_i}X_i}\tau_b(\varphi) \right\} \\
&= \sum_{i=1}^{2n} \left[ \sum_{j=1}^{2n} \bar{\nabla}_{X_i} \left\{ \langle \bar{\nabla}_X\tau_b(\varphi), d\varphi(X_j) \rangle d\varphi(X_j) \right\} \right. \\
&\quad \left. - \sum_{j=1}^{2n} \langle \bar{\nabla}_{\nabla_{X_i}X_i}\tau_b(\varphi), d\varphi(X_j) \rangle d\varphi(X_j) \right] \\
&= \sum_{i,j=1}^{2n} \left[ \begin{aligned} &\langle \bar{\nabla}_{X_i}(\bar{\nabla}_{X_i}\tau_b(\varphi)), d\varphi(X_j) \rangle d\varphi(X_j) \\ &+ \langle \bar{\nabla}_{X_i}\tau_b(\varphi), \bar{\nabla}_{X_i}(d\varphi(X_j)) \rangle d\varphi(X_j) \\ &+ \langle \bar{\nabla}_{X_i}\tau_b(\varphi), d\varphi(X_j) \rangle \bar{\nabla}_{X_i}d\varphi(X_j) \\ &- \langle \bar{\nabla}_{\nabla_{X_i}X_i}\tau_b(\varphi), d\varphi(X_j) \rangle d\varphi(X_j) \end{aligned} \right] \\
&= \sum_{j=1}^{2n} \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle d\varphi(X_j) \\
&\quad + \sum_{i,j=1}^{2n} \left[ \begin{aligned} &\langle \bar{\nabla}_{X_i}\tau_b(\varphi), \bar{\nabla}_{X_i}(d\varphi(X_j)) \rangle d\varphi(X_j) \\ &+ \langle \bar{\nabla}_{X_i}\tau_b(\varphi), d\varphi(X_j) \rangle \bar{\nabla}_{X_i}d\varphi(X_j) \end{aligned} \right]. \quad (5.7)
\end{aligned}$$

(The third step) Here, we have

$$\begin{cases} (\widetilde{\nabla}_{X_i}d\varphi)(X_j) = \bar{\nabla}_{X_i}d\varphi(X_j) - d\varphi(\nabla_{X_i}X_j) \in T^\perp M, \\ \bar{\nabla}_{X_i}\tau_b(\varphi) \in T^\perp M, \end{cases} \quad (5.8)$$

where  $\nabla$  is the Tanaka-Webster connection and  $\nabla_{X_i}X_j \in H(M)$ . Then, we have, in the first term of the second sum of (5.7),

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \langle \bar{\nabla}_{X_i}\tau_b(\varphi), \bar{\nabla}_{X_i}d\varphi(X_j) \rangle d\varphi(X_j) \\
&= \sum_{i,j=1}^{2n} \left\langle \bar{\nabla}_{X_i}\tau_b(\varphi), (\widetilde{\nabla}_{X_i}d\varphi)(X_j) + d\varphi(\nabla_{X_i}X_j) \right\rangle d\varphi(X_j) \\
&= \sum_{i,j=1}^{2n} \langle \bar{\nabla}_{X_i}\tau_b(\varphi), d\varphi(\nabla_{X_i}X_j) \rangle d\varphi(X_j) \\
&= \sum_{i,j=1}^{2n} \left\langle \bar{\nabla}_{X_i}\tau_b(\varphi), d\varphi \left( \sum_{k=1}^{2n} \langle \nabla_{X_i}X_j, X_k \rangle X_k \right) \right\rangle d\varphi(X_j), \quad (5.9)
\end{aligned}$$

because of  $\nabla_{X_i} X_j \in H(M)$ . Since the Tanaka-Webster connection  $\nabla$  satisfies  $\nabla g_\theta = 0$ , we have

$$\langle \nabla_{X_i} X_j, X_k \rangle = X_i \langle X_j, X_k \rangle - \langle X_j, \nabla_{X_i} X_k \rangle = -\langle X_j, \nabla_{X_i} X_k \rangle.$$

Thus, (5.9) turns to

$$\begin{aligned} & \sum_{i,j=1}^{2n} \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle d\varphi(X_j) \\ &= \sum_{i,j,k=1}^{2n} \langle \nabla_{X_i} X_j, X_k \rangle \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_k) \rangle d\varphi(X_j) \\ &= - \sum_{i,j,k=1}^{2n} \langle X_j, \nabla_{X_i} X_k \rangle \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_k) \rangle d\varphi(X_j) \\ &= - \sum_{i,k=1}^{2n} \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_k) \rangle d\varphi(\nabla_{X_i} X_k). \end{aligned} \quad (5.10)$$

(The forth step) By inserting (5.10) into (5.7), the second sum of (5.7) turns to

$$\begin{aligned} & \sum_{i,j=1}^{2n} \left[ \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} (d\varphi(X_j)) \rangle d\varphi(X_j) \right. \\ & \quad \left. + \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle \bar{\nabla}_{X_i} d\varphi(X_j) \right] \\ &= \sum_{i,j=1}^{2n} \left[ \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle \bar{\nabla}_{X_i} d\varphi(X_j) \right. \\ & \quad \left. - \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle d\varphi(\nabla_{X_i} X_j) \right] \\ &= \sum_{i,j=1}^{2n} \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle \left( \widetilde{\nabla}_{X_i} d\varphi \right) (X_j). \end{aligned} \quad (5.11)$$

Thus, by (5.7) and (5.11), we obtain Lemma 5.3.  $\square$

(*Proof of Lemma 5.4*) We will calculate the right hand side of (5.2) in Lemma 5.3. The proof is divided into several steps.

(The first step) We first note that

$$\begin{aligned} \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle + \langle \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle &= X_i \langle \tau_b(\varphi), d\varphi(X_j) \rangle \\ &= 0. \end{aligned} \quad (5.12)$$

Thus, by (5.12), we have

$$\begin{aligned}
\langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(X_j) \rangle &= -\langle \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle \\
&= -\langle \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) - d\varphi(\nabla_{X_i} X_j) \rangle \\
&= -\langle \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) - d\varphi(\nabla^{g\theta}_{X_i} X_j) \rangle \\
&\quad (\text{by } \langle \tau_b(\varphi), d\varphi(T) \rangle = 0) \\
&= -\langle \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(X_j) \rangle. \tag{5.13}
\end{aligned}$$

(The second step) By differentiating (5.12), we have

$$\begin{aligned}
\langle \bar{\nabla}_{X_i} (\bar{\nabla}_{X_i} \tau_b(\varphi)), d\varphi(X_j) \rangle &+ 2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle \\
&+ \langle \tau_b(\varphi), \bar{\nabla}_{X_i} (\bar{\nabla}_{X_i} d\varphi(X_j)) \rangle = 0. \tag{5.14}
\end{aligned}$$

And we have

$$\begin{aligned}
\langle \bar{\nabla}_{\nabla_{X_i} X_i} \tau_b(\varphi), d\varphi(X_j) \rangle &+ \langle \tau_b(\varphi), \bar{\nabla}_{\nabla_{X_i} X_i} d\varphi(X_j) \rangle \\
&= \nabla_{X_i} X_i \langle \tau_b(\varphi), d\varphi(X_j) \rangle = 0. \tag{5.15}
\end{aligned}$$

Thus, by (5.14) and (5.15), we have

$$\begin{aligned}
\langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle &+ 2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle \\
&+ \langle \tau_b(\varphi), -\Delta_b(d\varphi(X_j)) \rangle = 0. \tag{5.16}
\end{aligned}$$

(The third step) For the second term of the left hand side of (5.16), we have

$$2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle = -2 \langle \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(\nabla_{X_i} X_j) \rangle. \tag{5.17}$$

Because, the left hand side of (5.17) is

$$\begin{aligned}
2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle &= 2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(X_j) + d\varphi(\nabla^{g\theta}_{X_i} X_j) \rangle \\
&= 2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(X_j) + d\varphi(\nabla_{X_i} X_j) \rangle \\
&= 2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), d\varphi(\nabla_{X_i} X_j) \rangle \\
&= -2 \langle \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(\nabla_{X_i} X_j) \rangle \\
&\quad (\text{by } \langle \tau_b(\varphi), d\varphi(\nabla_{X_i} X_j) \rangle = 0) \\
&= -2 \langle \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(\nabla_{X_i} X_j) \rangle \tag{5.18}
\end{aligned}$$

which is the right hand side of (5.17). In the last step of (5.18), we used the equality  $\langle \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(T) \rangle = 0$  which follows from that  $\bar{\nabla}_{X_i} d\varphi(T)$  is tangential.

(The forth step) For the third term of the left hand side of (5.16), we have

$$\langle \tau_b(\varphi), -\Delta_b(d\varphi(X_j)) \rangle = \left\langle \tau_b(\varphi), \left( -\tilde{\Delta}_b d\varphi \right) (X_j) + 2 \sum_{k=1}^{2n} (\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \right\rangle. \quad (5.19)$$

Because, by the definition of  $\Delta_b$ , we have

$$\begin{aligned} & \langle \tau_b(\varphi), -\Delta_b(d\varphi(X_j)) \rangle \\ &= \langle \tau_b(\varphi), \sum_{k=1}^{2n} \left\{ \bar{\nabla}_{X_k} \left( \bar{\nabla}_{X_k} d\varphi(X_j) \right) - \bar{\nabla}_{\nabla_{X_k} X_k} d\varphi(X_j) \right\} \rangle \\ &= \left\langle \tau_b(\varphi), \sum_{k=1}^{2n} \left\{ \bar{\nabla}_{X_k} \left( (\tilde{\nabla}_{X_k} d\varphi)(X_j) + d\varphi(\nabla_{X_k} X_j) \right) \right. \right. \\ &\quad \left. \left. - \left( \tilde{\nabla}_{\nabla_{X_k} X_k} d\varphi \right)(X_j) - d\varphi(\nabla_{\nabla_{X_k} X_k} X_j) \right\} \right\rangle \\ &= \left\langle \tau_b(\varphi), \sum_{k=1}^{2n} \left\{ \left( \tilde{\nabla}_{X_k} \tilde{\nabla}_{X_k} d\varphi \right)(X_j) + \left( \tilde{\nabla}_{X_k} d\varphi \right)(\nabla_{X_k} X_j) \right. \right. \\ &\quad \left. \left. + \left( \tilde{\nabla}_{X_k} d\varphi \right)(\nabla_{X_k} X_j) + d\varphi(\nabla_{X_k} \nabla_{X_k} X_j) \right. \right. \\ &\quad \left. \left. - \left( \tilde{\nabla}_{\nabla_{X_k} X_k} d\varphi \right)(X_j) \right\} \right\rangle \\ &= \left\langle \tau_b(\varphi), \left( -\tilde{\Delta}_b d\varphi \right)(X_j) + 2 \sum_{k=1}^{2n} \left( \tilde{\nabla}_{X_k} d\varphi \right)(\nabla_{X_k} X_j) \right\rangle, \end{aligned}$$

which is (5.19). To get the last equality of the above, we used the following equations: for all  $X \in H(M)$ , it holds that

$$\langle \tau_b(\varphi), (\bar{\nabla}_X d\varphi)(T) \rangle = \langle \tau_b(\varphi), d\varphi(X) \rangle = \langle \tau_b(\varphi), (\tilde{\nabla}_X d\varphi)(T) \rangle = 0. \quad (5.20)$$

To get (5.20), due to the admissibility of  $\varphi$ , we have

$$(\tilde{\nabla}_X d\varphi)(T) = (\bar{\nabla}_X d\varphi)(T) - d\varphi(\nabla_X^{g_\theta} T) = B_\varphi(X, T) = 0,$$

and then,  $(\bar{\nabla}_X d\varphi)(T)$  is tangential for all  $X \in H(M)$ . We have (5.20), and then (5.19).

(The fifth step) Then, the right hand side of (5.19) is equal to

$$\begin{aligned}
& \langle \tau_b(\varphi), (-\tilde{\Delta}_b d\varphi)(X_j) + 2 \sum_{k=1}^{2n} (\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \rangle \\
&= \langle \tau_b(\varphi), \bar{\nabla}_{X_j} \tau(\varphi) \\
&\quad - \sum_{k=1}^{2n} \left\{ R^h(d\varphi(X_j), d\varphi(X_k)) d\varphi(X_k) - d\varphi(R^{g_\theta}(X_j, X_k) X_k) \right\} \\
&\quad - \sum_{k=1}^{2n} \left\{ R^h(d\varphi(X_j), d\varphi(T)) d\varphi(T) - d\varphi((R^{g_\theta}(X_j, T) T) \right\} \\
&\quad - \tilde{\nabla}_T \tilde{\nabla}_T d\varphi(X_j) \\
&\quad + 2(\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \rangle, \tag{5.21}
\end{aligned}$$

which follows from the formula (3.11).

Here, notice that

$$\langle \tau_b(\varphi), \bar{\nabla}_{X_j} \tau(\varphi) \rangle = 0. \tag{5.22}$$

Because  $\tau_b(\varphi)$  is normal, and  $\bar{\nabla}_{X_j} \tau(\varphi)$  is tangential. And also we have

$$\langle \tau_b(\varphi), \tilde{\nabla}_T(d\varphi(X_j)) \rangle = 0, \tag{5.23}$$

$$\langle \tau_b(\varphi), \tilde{\nabla}_T \tilde{\nabla}_T d\varphi(X_j) \rangle = 0. \tag{5.24}$$

To see (5.23), since we assume  $\varphi$  is an admissible isometric immersion, we have

$$\tilde{\nabla}_T d\varphi(X_j) = \nabla_T^h X_j = \nabla_T^{g_\theta} X_j + B_\varphi(T, X_j) = \nabla_T^{g_\theta} X_j \tag{5.25}$$

which is tangential, so that we have (5.23). Furthermore, to see (5.24), we have

$$\begin{aligned}
\tilde{\nabla}_T \tilde{\nabla}_T d\varphi(X_j) &= \tilde{\nabla}(\nabla_T^{g_\theta} X_j) \\
&= \nabla_T^h(\nabla_T^{g_\theta} X_j) \\
&= \nabla_T^{g_\theta}(\nabla_T^{g_\theta} X_j) + B(T, \nabla_T^{g_\theta} X_j). \tag{5.26}
\end{aligned}$$

Here, for every  $X \in H(M)$ ,

$$\nabla_T^{g_\theta} X \in H(M).$$

Indeed, since  $g_\theta(T, X) = 0$ , and  $\nabla_T^{g_\theta} T = 0$  (cf. [11], pp. 47, and 48),

$$g_\theta(T, \nabla_T^{g_\theta} X) = T(g_\theta(T, X)) - g_\theta(\nabla_T^{g_\theta} T, X) = 0,$$

which implies  $\nabla_T^{g_\theta} X \in H(M)$ . Thus, the admissibility implies that the second term of (5.26) vanishes. Thus, the right hand side of (5.26) is tangential, which implies (5.24).

Therefore, we obtain

$$\begin{aligned}
& \langle \tau_b(\varphi), (-\tilde{\Delta}_b d\varphi)(X_j) + 2 \sum_{k=1}^{2n} (\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \rangle \\
&= - \sum_{k=1}^{2n} \langle \tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(X_k)) d\varphi(X_k) \rangle \\
&\quad - \sum_{k=1}^{2n} \langle \tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(T)) d\varphi(T) \rangle \\
&\quad + 2 \sum_{k=1}^{2n} \langle \tau_b(\varphi), (\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \rangle. \tag{5.27}
\end{aligned}$$

(The sixth step) Now, return to (5.16), by using (5.17), (5.19) and (5.27), we have

$$\begin{aligned}
0 &= \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle + 2 \langle \bar{\nabla}_{X_i} \tau_b(\varphi), \bar{\nabla}_{X_i} d\varphi(X_j) \rangle \\
&\quad + \langle \tau_b(\varphi), -\Delta_b(d\varphi(X_j)) \rangle \\
&= \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle - 2 \langle \tau_b(\varphi), (\tilde{\nabla}_{X_i} d\varphi)(\nabla_{X_i} X_j) \rangle \\
&\quad + \langle \tau_b(\varphi), -\Delta_b(d\varphi(X_j)) \rangle \\
&= \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle - 2 \langle \tau_b(\varphi), (\tilde{\nabla}_{X_i} d\varphi)(\nabla_{X_i} X_j) \rangle \\
&\quad + \langle \tau_b(\varphi), (-\tilde{\Delta}_b d\varphi)(X_j) + 2 \sum_{k=1}^{2n} (\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \rangle \\
&= \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle - 2 \langle \tau_b(\varphi), (\tilde{\nabla}_{X_i} d\varphi)(\nabla_{X_i} X_j) \rangle \\
&\quad + \left\langle \tau_b(\varphi), - \sum_{k=1}^{2n} R^h(d\varphi(X_j), d\varphi(X_k)) d\varphi(X_k) \right. \\
&\quad \quad \left. - R^h(d\varphi(X_j), d\varphi(T)) d\varphi(T) \right. \\
&\quad \quad \left. + 2 \sum_{k=1}^{2n} (\tilde{\nabla}_{X_k} d\varphi)(\nabla_{X_k} X_j) \right\rangle \\
&= \langle -\Delta_b(\tau_b(\varphi)), d\varphi(X_j) \rangle \\
&\quad + \left\langle \tau_b(\varphi), - \sum_{k=1}^{2n} R^h(d\varphi(X_j), d\varphi(X_k)) d\varphi(X_k) \right. \\
&\quad \quad \left. - R^h(d\varphi(X_j), d\varphi(T)) d\varphi(T) \right\rangle. \tag{5.28}
\end{aligned}$$

(The seventh step) Inserting (5.28) into (5.2) of Lemma 5.3, we obtain

$$\begin{aligned} -\Delta_b(\tau_b(\varphi)) &= \langle \tau_b(\varphi), \sum_{k=1}^{2n} R^h(d\varphi(X_j), d\varphi(X_k))d\varphi(X_k) \rangle d\varphi(X_j) \\ &\quad + \langle \tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(T))d\varphi(T) \rangle d\varphi(X_j) \\ &\quad + \langle \bar{\nabla}_{X_i}\tau_b(\varphi), d\varphi(X_j) \rangle (\bar{\nabla}_{X_i}d\varphi)(X_j). \end{aligned} \quad (5.29)$$

At last, for the third term of (5.29), we have

$$\begin{aligned} \langle \bar{\nabla}_{X_i}\tau_b(\varphi), d\varphi(X_j) \rangle &= X_i \langle \tau_b(\varphi), d\varphi(X_j) \rangle - \langle \tau_b(\varphi), \bar{\nabla}_{X_i}d\varphi(X_j) \rangle \\ &= -\langle \tau_b(\varphi), (\bar{\nabla}_{X_i}d\varphi)(X_j) \rangle. \end{aligned} \quad (5.30)$$

Together with (5.29) and (5.30), we have (5.3) of Lemma 5.4.  $\square$

Due to Lemma 5.4 and the definition of biharmonicity, we obtain immediately

**Theorem 5.5.** *Let  $\varphi$  be an admissible isometric immersion of a strongly pseudoconvex CR manifold  $(M, g_\theta)$  into another Riemannian manifold  $(N, h)$  whose pseudo mean curvature vector field along  $\varphi$  is parallel. Then,  $\varphi$  is pseudo biharmonic if and only if*

$$\tau_{b,2}(\varphi) := \Delta_b(\tau_b(\varphi)) - \sum_{j=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_j))d\varphi(X_j) = 0 \quad (5.31)$$

if and only if

$$\begin{aligned} & - \sum_{j,k=1}^{2n} h(\tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(X_k))d\varphi(X_k)) d\varphi(X_j) \\ & - \sum_{j=1}^{2n} h(\tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(T))d\varphi(T)) d\varphi(X_j) \\ & + \sum_{j,k=1}^{2n} h(\tau_b(\varphi), B_\varphi(X_j, X_k)) B_\varphi(X_j, X_k) \\ & - \sum_{j=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_j))d\varphi(X_j) = 0, \end{aligned} \quad (5.32)$$

where  $\{X_j\}_{j=1}^{2n}$  is an orthonormal frame field of  $(H(M), g_\theta)$ .

**Remark 5.6.** Due to [20], p. 220, Lemma 10, and the definition of bi-tension field  $\tau_2(\varphi)$  for an isometric immersion  $\varphi$  of a Riemannian manifold  $(M, g)$  into another Riemannian manifold  $(N, h)$ , we can also obtain immediately the following useful theorem:



**Theorem** *Let  $\varphi$  be an isometric immersion of a Riemannian manifold  $(M^m, g)$  into another Riemannian manifold  $(N^n, h)$  whose mean curvature vector field along  $\varphi$  is parallel. Let  $\{e_j\}_{j=1}^m$  be an orthonormal frame field of  $(M, g)$ . Then,  $\varphi$  is biharmonic if and only if*

$$\tau_2(\varphi) := \overline{\Delta}(\tau(\varphi)) - \sum_{j=1}^m R^h(\tau(\varphi), e_j) e_j = 0 \quad (5.33)$$

*if and only if*

$$\begin{aligned} & - \sum_{j,k=1}^m h(\tau(\varphi), R^h(d\varphi(e_j), d\varphi(e_k)) d\varphi(e_k)) d\varphi(e_j) \\ & + \sum_{j,k=1}^m h(\tau(\varphi), B_\varphi(e_j, e_k)) B_\varphi(e_j, e_k) \\ & - \sum_{j=1}^m R^h(\tau(\varphi), d\varphi(e_j)) d\varphi(e_j) = 0. \end{aligned} \quad (5.34)$$

## 6. ISOMETRIC IMMERSIONS INTO THE UNIT SPHERE.

In this section, we treat with admissible isometric immersions of  $(M^{2n+1}, g_\theta)$  into the unit sphere  $(N, h) = S^{2n+2}(1)$  with parallel pseudo mean curvature vector field with non-zero pseudo mean curvature.

The curvature tensor field  $R^h$  of the target space  $(N, h) = S^{2n+2}(1)$  satisfies that

$$R^h(X, Y)Z = h(Z, Y)X - h(Z, X)Y \quad (6.1)$$

for all vector fields  $X, Y, Z$  on  $N$ . Then, we have

$$(R^h(d\varphi(X_j), d\varphi(X_k)) d\varphi(X_k))^\perp = 0, \quad (6.2)$$

$$(R^h(d\varphi(X_j), d\varphi(T)) d\varphi(T))^\perp = 0, \quad (6.3)$$

for all  $i, j = 1, \dots, 2n$ . Therefore, we obtain by (5.3) in Lemma 5.4,

$$-\Delta_b(\tau_b(\varphi)) = -\langle \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(X_j) \rangle (\widetilde{\nabla}_{X_i} d\varphi)(X_j). \quad (6.4)$$

On the other hand, we have

$$\begin{aligned} & \sum_{k=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_k)) d\varphi(X_k) \\ & = \sum_{k=1}^{2n} h(d\varphi(X_k), d\varphi(X_k)) \tau_b(\varphi) - \sum_{k=1}^{2n} h(d\varphi(X_k), \tau_b(\varphi)) \tau_b(\varphi) \\ & = 2n \tau_b(\varphi). \end{aligned} \quad (6.5)$$

Now, let us recall the pseudo biharmonicity of  $\varphi$  is equivalent to that

$$-\Delta_b(\tau_b(\varphi)) + \sum_{k=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_k)) d\varphi(X_k) = 0 \quad (6.6)$$

which is equivalent to that

$$-\langle \tau_b(\varphi), (\widetilde{\nabla}_{X_i} d\varphi)(X_j) \rangle (\widetilde{\nabla}_{X_i} d\varphi)(X_j) + 2n \tau_b(\varphi) = 0. \quad (6.7)$$

For our immersion  $\varphi : (M, g_\theta) \rightarrow S^{2n+2}(1)$ , let  $\xi$  be the unit normal vector field on  $M$  along  $\varphi$ , we have by definition,

$$(\widetilde{\nabla}_{X_i} d\varphi)(X_j) = B_\varphi(X_i, X_j) = H_{ij} \xi. \quad (6.8)$$

Then, we have by definition of  $\tau_b(\varphi)$ ,

$$\tau_b(\varphi) = \sum_{i=1}^{2n} (\widetilde{\nabla}_{X_i} d\varphi)(X_i) = \left( \sum_{i=1}^{2n} H_{ii} \right) \xi. \quad (6.9)$$

Therefore, we have

$$\|\tau_b(\varphi)\|^2 = \left( \sum_{i=1}^{2n} H_{ii} \right)^2 \|\xi\|^2 = \left( \sum_{i=1}^{2n} H_{ii} \right)^2. \quad (6.10)$$

By the admissibility, we have

$$\begin{aligned} \|B_\varphi\|^2 &= \sum_{i,j=1}^{2n} \|B_\varphi(X_i, X_j)\|^2 + 2 \sum_{i=1}^{2n} \|B_\varphi(X_i, T)\|^2 + \|B_\varphi(T, T)\|^2 \\ &= \sum_{i,j=1}^{2n} \|H_{ij} \xi\|^2 + \|B_\varphi(T, T)\|^2 \\ &= \sum_{i,j=1}^{2n} H_{ij}^2 + \|B_\varphi(T, T)\|^2. \end{aligned} \quad (6.11)$$

Due to (6.7), (6.8) and (6.9), the biharmonicity of  $\varphi$  is equivalent to that

$$0 = - \left\langle \left( \sum_{k=1}^{2n} H_{kk} \right) \xi, H_{ij} \xi \right\rangle H_{ij} \xi + 2n \left( \sum_{k=1}^{2n} H_{kk} \right) \xi \quad (6.12)$$

which is equivalent to that

$$\begin{aligned} 0 &= \left( \sum_{k=1}^{2n} H_{kk} \right) \left\{ - \sum_{i,j=1}^{2n} H_{ij}^2 + 2n \right\} \\ &= \|\tau_b(\varphi)\| \left\{ -\|B_\varphi\|^2 + \|B_\varphi(T, T)\|^2 + 2n \right\} \end{aligned} \quad (6.13)$$

by (6.11). By our assumption of non-zero pseudo mean curvature,  $\|\tau_b(\varphi)\| \neq 0$  at every point, we obtain the following equivalence relation:  $\varphi$  is pseudo biharmonic if and only if

$$\|B_\varphi\|^2 = \|B_\varphi(T, T)\|^2 + 2n \quad (6.14)$$

at every point in  $M$ .

By summing up the above, we obtain the following theorem:

**Theorem 6.1.** *Let  $\varphi$  be an admissible isometric immersion of a strictly pseudoconvex CR manifold  $(M, g_\theta)$  into the unit sphere  $(N, h) = S^{2n+2}(1)$ . Assume that the pseudo mean curvature vector field is parallel with non-zero pseudo mean curvature. Then,  $\varphi$  is pseudo biharmonic if and only if*

$$\|B_\varphi\|^2 = \|B_\varphi(T, T)\|^2 + 2n. \quad (6.15)$$

The admissibility condition is that:  $d\varphi(T)$  is the principal curvature vector field along  $\varphi$  with some principal curvature, say  $\lambda_{2n+1}$ . I.e.,

$$A_\xi T = \lambda_{2n+1} T.$$

Then, we have

$$\|B_\varphi\|^2 = \sum_{i=1}^{2n+1} \lambda_i^2, \quad \text{and} \quad \|B_\varphi(T, T)\|^2 = \lambda_{2n+1}^2.$$

By Theorem 6.1, we have immediately Thus, we obtain

**Corollary 6.2.** *Let  $\varphi : (M^{2n+1}, g_\theta) \rightarrow S^{2n+2}(1)$  be an isometric immersion whose the pseudo mean curvature vector field is parallel and has non-zero pseudo mean curvature. Then,  $\varphi$  is pseudo biharmonic if and only if it holds that*

$$\sum_{i=1}^{2n} \lambda_i^2 = 2n \quad (6.16)$$

which is equivalent to that

$$\left\| B_\varphi \Big|_{H(M) \times H(M)} \right\|^2 = 2n, \quad (6.17)$$

where  $B_\varphi|_{H(M) \times H(M)}$  is the restriction of  $B_\varphi$  to the subspace  $H(M)$  of the tangent space  $T_x M$  ( $x \in M$ ).

## 7. ISOMETRIC IMMERSIONS TO THE COMPLEX PROJECTIVE SPACE.

In this section, we will consider admissible isometric immersions of  $(M^{2n+1}, g_\theta)$  into the complex projective space  $(N, h) = \mathbb{P}^{n+1}(c)$  ( $c > 0$ ) whose mean curvature vector field is parallel with non-zero pseudo mean curvature.

**7.1.** Let us recall that the curvature tensor field  $(N, h) = \mathbb{P}^{n+1}(c)$  is given by

$$R^h(U, V)W = \frac{c}{4} \left\{ h(V, W)U - h(U, W)V \right. \\ \left. + h(JV, W)JU - h(JU, W)JV + 2h(U, JV)JW \right\}, \quad (7.1)$$

where  $J$  is the adapted complex structure of  $\mathbb{P}^{n+1}(c)$ , and  $U, V$  and  $W$  are vector fields on  $\mathbb{P}^{n+1}(c)$ , respectively. Therefore, we have

$$R^h(d\varphi(X_j), d\varphi(X_k))d\varphi(X_k) \\ = \frac{c}{4} \left\{ h(d\varphi(X_k), d\varphi(X_k))d\varphi(X_j) - h(d\varphi(X_j), d\varphi(X_k))d\varphi(X_k) \right. \\ \left. + h(Jd\varphi(X_k), d\varphi(X_k))Jd\varphi(X_j) - h(Jd\varphi(X_j), d\varphi(X_k))Jd\varphi(X_k) \right. \\ \left. + 2h(d\varphi(X_j), Jd\varphi(X_k))Jd\varphi(X_k) \right\} \\ = \frac{c}{4} \left\{ d\varphi(X_j) - \delta_{jk}d\varphi(X_k) + 3h(d\varphi(X_j), Jd\varphi(X_k))Jd\varphi(X_k) \right\}. \quad (7.2)$$

We show first

$$\sum_{j,k=1}^{2n} h\left(\tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(X_k))d\varphi(X_k)\right)d\varphi(X_j) \\ = -\frac{3c}{4} h(\tau_b(\varphi), Jd\varphi(T))\left(Jd\varphi(T)\right)^\top \\ - \frac{3c}{4} \sum_{j=1}^{2n} h\left(d\varphi(X_j), J\left(J\tau_b(\varphi)\right)^\top\right)d\varphi(X_j). \quad (7.3)$$

Recall here that the tangential part of  $Z \in T_{\varphi(x)}N$  ( $x \in M$ ) is given by

$$Z^\top = \sum_{i=1}^{2n} h(Z, d\varphi(X_i))d\varphi(X_i) + h(Z, d\varphi(T))d\varphi(T). \quad (7.4)$$

Since  $h(\tau_b(\varphi), d\varphi(X_j)) = 0$  ( $j = 1, \dots, 2n$ ), and (7.2), one can calculate the left hand side of (7.3) as follows:

$$\begin{aligned}
& \sum_{j,k=1}^{2n} h\left(\tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(X_k)) d\varphi(X_k)\right) d\varphi(X_j) \\
&= \frac{3c}{4} \sum_{j,k=1}^{2n} h(\tau_b(\varphi), h(d\varphi(X_j), J d\varphi(X_k)) h(\tau_b(\varphi), J d\varphi(X_k)) d\varphi(X_j) \\
&= \frac{3c}{4} \sum_{j,k=1}^{2n} h(J d\varphi(X_j), d\varphi(X_k)) h(J \tau_b(\varphi), d\varphi(X_k)) d\varphi(X_j) \\
&= \frac{3c}{4} \sum_{j=1}^{2n} h\left(J d\varphi(X_j), \sum_{k=1}^{2n} h(J \tau_b(\varphi), d\varphi(X_k)) d\varphi(X_k)\right) d\varphi(X_j) \\
&= \frac{3c}{4} \sum_{j=1}^{2n} h\left(J d\varphi(X_j), (J \tau_b(\varphi))^\top - h(J \tau_b(\varphi), d\varphi(T)) d\varphi(T)\right) d\varphi(X_j) \\
&= \frac{3c}{4} \sum_{j=1}^{2n} h(J d\varphi(X_j), (J \tau_b(\varphi))^\top) d\varphi(X_j) \\
&\quad + \frac{3c}{4} h(J \tau_b(\varphi), d\varphi(T)) \sum_{j=1}^{2n} h(d\varphi(X_j), J d\varphi(T)) d\varphi(X_j) \\
&= -\frac{3c}{4} \sum_{j=1}^{2n} h(d\varphi(X_j), J (J \tau_b(\varphi))^\top) d\varphi(X_j) \\
&\quad - \frac{3c}{4} h(\tau_b(\varphi), J d\varphi(T)) (J d\varphi(T))^\top. \tag{7.5}
\end{aligned}$$

Then, (7.5) is just (7.3).

Second, by a similar way,

$$\begin{aligned}
& \sum_{j=1}^{2n} \langle \tau_b(\varphi), R^h(d\varphi(X_j), d\varphi(T)) d\varphi(T) \rangle d\varphi(X_j) \\
&= \sum_{j=1}^{2n} \langle \tau_b(\varphi), \frac{3c}{4} h(d\varphi(X_j), J d\varphi(T)) J d\varphi(T) \rangle d\varphi(X_j) \\
&= \frac{3c}{4} h(\tau_b(\varphi), J d\varphi(T)) (J d\varphi(T))^\top \tag{7.6}
\end{aligned}$$

in the last equality of (7.6) we used that  $h(d\varphi(T), J d\varphi(T)) = 0$ .

Thus, we have

$$\begin{aligned}
\Delta_b(\tau_b(\varphi)) &= \frac{3c}{4} h(\tau_b(\varphi), Jd\varphi(T)) (Jd\varphi(T))^\top \\
&\quad + \frac{3c}{4} \sum_{j=1}^{2n} h(d\varphi(X_j), J(J\tau_b(\varphi))^\top) d\varphi(X_j) \\
&\quad - \frac{3c}{4} h(\tau_b(\varphi), Jd\varphi(T)) (Jd\varphi(T))^\top \\
&\quad + \langle \tau_b(\varphi), B_\varphi(X_i, X_j) \rangle B_\varphi(X_i, X_j) \\
&= \frac{3c}{4} \sum_{j=1}^{2n} h(d\varphi(X_j), J(J\tau_b(\varphi))^\top) d\varphi(X_j) \\
&\quad + \langle \tau_b(\varphi), B_\varphi(X_i, X_j) \rangle B_\varphi(X_i, X_j). \tag{7.7}
\end{aligned}$$

Therefore, an isometric immersion  $\varphi$  is pseudo biharmonic if and only if the pseudo biharmonic map equation folds:

$$\Delta_b(\tau_b(\varphi)) - \sum_{k=1}^{2n} R^h(\tau_b(\varphi), d\varphi(X_k) d\varphi(X_k)) = 0. \tag{7.8}$$

By (7.7) and (7.1), (7.8) is equivalent to that the following (7.9) holds:

$$\begin{aligned}
&\frac{3c}{4} \sum_{j=1}^{2n} h(d\varphi(X_j), J(J\tau_b(\varphi))^\top) d\varphi(X_j) \\
&\quad + \langle \tau_b(\varphi), B_\varphi(X_i, X_j) \rangle B_\varphi(X_i, X_j) \\
&\quad - \frac{2nc}{4} \tau_b(\varphi) + \frac{3c}{4} \sum_{k=1}^{2n} h(d\varphi(X_j), J\tau_b(\varphi)) Jd\varphi(X_k) \\
&= 0. \tag{7.9}
\end{aligned}$$

**7.2.** Let  $\xi$  be the unit normal vector field along the admissible isometric immersion  $\varphi : (M, g_\theta) \rightarrow \mathbb{P}^{n+1}(c)$  ( $c > 0$ ).

We have immediately

$$\begin{cases} B_\varphi(X_i, X_i) = (\widetilde{\nabla}_{X_i} d\varphi)(X_j) = H_{ij} \xi, \\ \tau_b(\varphi) = \sum_{k=1}^{2n} (\widetilde{\nabla}_{X_k} d\varphi)(X_k) = \left( \sum_{k=1}^{2n} H_{kk} \right) \xi, \\ J\tau_b(\varphi) = \left( \sum_{k=1}^{2n} H_{kk} \right) J\xi, \end{cases} \tag{7.10}$$

and then, we have

$$h(\xi, J\xi) = 0. \tag{7.11}$$

Indeed, we have

$$-h(J\xi, \xi) = h(J\xi, J(J\xi)) = h(\xi, J\xi),$$

which implies that  $h(\xi, J\xi) = 0$ .

Due to (7.11),  $J\xi$  is tangential. By (7.10),  $J\tau_b(\varphi)$  is also tangential. Therefore, we have

$$(J\tau_b(\varphi))^\top = J\tau_b(\varphi). \quad (7.12)$$

In particular, we have

$$\begin{aligned} & \sum_{j=1}^{2n} h(d\varphi(X_j), J(J\tau_b(\varphi))^\top) d\varphi(X_j) \\ &= \sum_{j=1}^{2n} h(d\varphi(X_j), J(J\tau_b(\varphi))) d\varphi(X_j) \\ &= - \sum_{j=1}^{2n} h(d\varphi(X_j), \tau_b(\varphi)) d\varphi(X_j) \\ &= 0 \end{aligned} \quad (7.13)$$

by using (7.12) and  $\tau_b(\varphi)$  is a normal vector field along  $\varphi$ .

Since  $J\tau_b(\varphi)$  is tangential, we can write as

$$J\tau_b(\varphi) = \sum_{k=1}^{2n} h(d\varphi(X_k), J\tau_b(\varphi)) d\varphi(X_k) + h(d\varphi(T), J\tau_b(\varphi)) d\varphi(T),$$

which implies that

$$\begin{aligned} & \sum_{k=1}^{2n} h(d\varphi(X_k), J\tau_b(\varphi)) d\varphi(X_k) \\ &= J\tau_b(\varphi) - h(d\varphi(T), J\tau_b(\varphi)) d\varphi(T). \end{aligned} \quad (7.14)$$

Therefore, applying  $J$  to (7.14), we have

$$\begin{aligned} & \sum_{k=1}^{2n} h(d\varphi(X_k), J\tau_b(\varphi)) J d\varphi(X_k) \\ &= J^2 \tau_b(\varphi) - h(d\varphi(T), J\tau_b(\varphi)) J d\varphi(T) \\ &= -\tau_b(\varphi) - h(d\varphi(T), J\tau_b(\varphi)) J d\varphi(T). \end{aligned} \quad (7.15)$$

Inserting (7.15) into (7.9), the left hand side of (7.9) is equal to

$$\begin{aligned}
& \frac{3c}{4} \sum_{j=1}^{2n} h(d\varphi(X_j), J((J\tau_b(\varphi))^\top)) d\varphi(X_j) \\
& + \sum_{i,j=1}^{2n} h(\tau_b(\varphi), B_\varphi(X_i, X_j) B_\varphi(X_i, X_j) \\
& - \frac{2nc}{4} \tau_b(\varphi) + \frac{3c}{4} \left\{ -\tau_b(\varphi) - h(d\varphi(T), J\tau_b(\varphi)) Jd\varphi(T) \right\} \\
& = \sum_{i,j=1}^{2n} h(\tau_b(\varphi), B_\varphi(X_i, X_j)) B_\varphi(X_i, X_j) \\
& - \frac{c(2n+3)}{4} \tau_b(\varphi) - \frac{3c}{4} h(d\varphi(T), J\tau_b(\varphi)) Jd\varphi(T),
\end{aligned} \tag{7.16}$$

where we used (7.13) for vanishing the first term of the left hand side of (7.16).

Due to (7.9) and (7.16), we obtain the equivalence relation that  $\varphi$  is biharmonic if and only if both the equations

$$(1) \quad h(d\varphi(T), J\tau_b(\varphi)) (Jd\varphi(T))^\top = 0, \tag{7.17}$$

and

$$\begin{aligned}
(2) \quad & \sum_{i,j=1}^{2n} h(\tau_b(\varphi), B_\varphi(X_i, X_j)) B_\varphi(X_i, X_j) - \frac{c(2n+3)}{4} \tau_b(\varphi) \\
& - \frac{3c}{4} h(d\varphi(T), J\tau_b(\varphi)) (Jd\varphi(T))^\perp = 0,
\end{aligned} \tag{7.18}$$

hold.

**7.3** For the first equation (1) (7.17) is equivalent to that

$$h(d\varphi(T), J\tau_b(\varphi)) = 0 \quad \text{or} \quad (Jd\varphi(T))^\top = 0. \tag{7.19}$$

But, by (7.10), we have

$$\begin{aligned}
h(d\varphi(T), J\tau_b(\varphi)) &= \left( \sum_{k=1}^{2n} H_{kk} \right) h(d\varphi(T), J\xi) \\
&= - \left( \sum_{k=1}^{2n} H_{kk} \right) h(Jd\varphi(T), \xi).
\end{aligned} \tag{7.20}$$

By our assumption that the pseudo mean curvature  $\sum_{k=1}^{2n} H_{kk} \neq 0$ , to hold that  $h(d\varphi(T), J\tau_b(\varphi)) = 0$  is equivalent to that

$$h(Jd\varphi(T), \xi) = 0. \tag{7.21}$$



And to hold that  $(J d\varphi(T))^\top = 0$  is equivalent to that

$$J d\varphi(T) = h(J d\varphi(T), \xi) \xi. \quad (7.22)$$

Thus, (1) (7.17) holds if and only if

$$(7.21) \quad h(J d\varphi(T), \xi) = 0, \quad \text{or}$$

$$(7.22) \quad J d\varphi(T) = h(J d\varphi(T), \xi) \xi.$$

In the case (7.21) holds, we have

$$h(d\varphi(T), J \tau_b(\varphi)) (J d\varphi(T))^\top = 0, \quad (7.23)$$

which implies that (2) (7.18) turns out that

$$\sum_{i,j=1}^{2n} h(\tau_b(\varphi), B_\varphi(X_i, X_j) B_\varphi(X_i, X_j) - \frac{c(2n+3)}{4} \tau_b(\varphi)) = 0. \quad (7.24)$$

In the case that (7.22) holds, we have that

$$\begin{aligned} & h(d\varphi(T), J \tau_b(\varphi)) (J d\varphi(T))^\top \\ &= h(d\varphi(T), J \tau_b(\varphi)) h(J d\varphi(T), \xi) \xi \\ &= \left( \sum_{k=1}^{2n} H_{kk} \right) h(d\varphi(T), J \xi) h(J d\varphi(T), \xi) \xi \quad (\text{by (7.10)}) \\ &= - \left( \sum_{k=1}^{2n} H_{kk} \right) h(J d\varphi(T), \xi)^2 \xi. \end{aligned} \quad (7.25)$$

In the case that (7.21) holds, (2) (7.18) turns out that

$$\begin{aligned} & \sum_{i,j=1}^{2n} h(\tau_b(\varphi), B_\varphi(X_i, X_j)) - \frac{c(2n+3)}{4} \tau_b(\varphi) \\ &+ \frac{3c}{4} \left( \sum_{k=1}^{2n} H_{kk} \right) h(J d\varphi(T), \xi)^2 \xi = 0. \end{aligned} \quad (7.26)$$

By inserting (7.10) into (7.24), the left hand side of (7.24) is equal to

$$\begin{aligned} & \sum_{i,j=1}^{2n} h \left( \sum_{k=1}^{2n} H_{kk} \xi, H_{ij} \xi \right) H_{ij} \xi - \frac{c(2n+3)}{4} \sum_{k=1}^{2n} H_{kk} \xi \\ &= \left( \sum_{k=1}^{2n} H_{kk} \right) \left\{ \sum_{i,j=1}^{2n} H_{ij}^2 - \frac{c(2n+3)}{4} \right\} \xi. \end{aligned} \quad (7.27)$$

(2) (7.18) is equivalent to that

$$\sum_{i,j=1}^{2n} H_{ij}^2 = \frac{c(2n+3)}{4} \quad (7.28)$$

by our assumption that  $\sum_{k=1}^{2n} H_{kk} \neq 0$ .

In the case that (7.22) holds, by inserting (7.10) into (7.26), the left hand side of (7.26) is equal to

$$\begin{aligned} & \sum_{i,j=1}^{2n} h\left(\sum_{k=1}^{2n} H_{kk} \xi, H_{ij} \xi\right) H_{ij} \xi - \frac{c(2n+3)}{4} \sum_{k=1}^{2n} H_{kk} \xi \\ & \quad + \frac{3c}{4} h(J d\varphi(X), \xi)^2 \left(\sum_{k=1}^{2n} H_{kk}\right) \xi \\ & = \left(\sum_{k=1}^{2n} H_{kk}\right) \left\{ \sum_{i,j=1}^{2n} H_{ij}^2 - \frac{c(2n+3)}{4} + \frac{3c}{4} h(J d\varphi(T), \xi)^2 \right\} \xi. \end{aligned} \quad (7.29)$$

Since (7.22)  $J d\varphi(T) = h(J d\varphi(T), \xi) \xi$ , we have

$$\begin{aligned} h(J d\varphi(T), \xi)^2 & = h(J d\varphi(T), d\varphi(T)) \\ & = h(d\varphi(T), d\varphi(T)) = g_\theta(T, T) = 1 \end{aligned}$$

which implies again by our assumption  $\sum_{k=1}^{2n} H_{kk} \neq 0$ , that (2) (7.18) is equivalent to that

$$\sum_{i,j=1}^{2n} H_{ij}^2 - \frac{c(2n+3)}{4} + \frac{3c}{4} = \sum_{i,j=1}^{2n} H_{ij}^2 - \frac{n}{2} c = 0. \quad (7.30)$$

Therefore, we obtain

**Theorem 7.1.** *Assume that  $\varphi : (M, g_\theta) \rightarrow \mathbb{P}^{n+1}(c) = (N, h)$  ( $c > 0$ ) is an admissible isometric immersion whose pseudo mean curvature vector field along  $\varphi$  is parallel with non-zero pseudo mean curvature. Then,  $\varphi$  is biharmonic if and only if one of the following two cases occurs:*

$$(1) \quad h(J d\varphi(T), \xi) = 0 \text{ and}$$

$$\left\| B_\varphi \Big|_{H(M) \times H(M)} \right\|^2 = \frac{c(2n+3)}{4}, \quad (7.31)$$

$$(2) \quad J d\varphi(T) = h(J d\varphi(T), \xi) \xi \text{ and}$$

$$\left\| B_\varphi \Big|_{H(M) \times H(M)} \right\|^2 = \frac{n}{2} c. \quad (7.32)$$

## 8. EXAMPLES OF PSEUDO HARMONIC MAPS AND PSEUDO BIHARMONIC MAPS

In this section, we give some examples of pseudo biharmonic maps.

*Example 8.1.* Let  $(M^{2n+1}, g_\theta) = S^{2n+1}(r)$  be the sphere of radius  $r$  ( $0 < r < 1$ ) which is embedded in the unit sphere  $S^{2n+2}(1)$ , i.e., the natural embedding  $\varphi : S^{2n+1}(r) \rightarrow S^{2n+2}(1)$  is given by

$$\varphi : S^{2n+1}(r) \ni x' = (x_1, x_2, \dots, x_{2n+2}) \mapsto (x', \sqrt{1-r^2}) \in S^{2n+2}(1).$$

This  $\varphi$  is a standard isometric with constant principal curvature  $\lambda_1 = \cot[\cos^{-1} t]$ , ( $-1 < t < 1$ ), with the multiplicity  $m_1 = \dim M = 2n+1$ .

Due to Theorem 6.2, it is *pseudo biharmonic* if and only if

$$\begin{aligned} (\lambda_1)^2 \times 2n = 2n & \Leftrightarrow \lambda_1 = \cot[\cos^{-1} t] = \pm 1. \\ & \Leftrightarrow t = \cos\left(\pm \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}. \end{aligned} \quad (8.1)$$

This is just the example which is biharmonic but not minimal given by C. Oniciuc ([29]). Note that  $\varphi : S^{2n+1}(r) \rightarrow S^{2n+2}(1)$  is *pseudo harmonic* if and only if

$$\begin{aligned} \text{Trace}(B_\varphi|_{H(M) \times H(M)}) = 0 & \Leftrightarrow \lambda_1 = 0 \\ & \Leftrightarrow t = \cos\left(\frac{\pi}{2}\right) = 1. \end{aligned} \quad (8.2)$$

This  $t = 1$  gives a great hypersphere which is also minimal.

*Example 8.2.* Let the Hopf fibration  $\pi : S^{2n+3}(1) \rightarrow \mathbb{P}^{n+1}(4)$ , and, let  $\widehat{M} := S^1(\cos u) \times S^{2n+1}(\sin u) \subset S^{2n+3}(1)$  ( $0 < u < \frac{\pi}{2}$ ). Then, we have  $\varphi : M^{2n+1} = \pi(\widehat{M}) \subset \mathbb{P}^{n+1}(4)$  which is a homogeneous real hypersurface of  $\mathbb{P}^{n+1}(4)$  of type  $A_1$  in the table of R. Takagi ([31]) whose principal curvatures and their multiplicities are given as follows ([31]):

$$\begin{cases} \lambda_1 = \cot u, & \text{multiplicity } m_1 = 2n, \\ \lambda_2 = 2 \cot(2u), & \text{multiplicity } m_2 = 1. \end{cases} \quad (8.3)$$

Since  $2 \cot(2u) = \cot u - \tan u$ , the mean curvature  $H$  and  $\|B_\varphi\|^2$  are given by

$$H = \frac{1}{2n+1} \{(2n+1) \cot u - \tan u\}, \quad (8.4)$$

$$\|B_\varphi\|^2 = m_1 \lambda_1^2 + m_2 \lambda_2^2 = \tan^2 u + (2n+1) \cot^2 u - 2. \quad (8.5)$$

R. Takagi showed ([31]) to this example, that  $\varphi : M^{2n+1} \rightarrow \mathbb{P}^{n+1}(4)$  is the geodesic sphere  $S^{2n+1}$ , and  $J(-\xi)$  is the mean curvature vector of the principal curvature  $\lambda_2$  (cf. Remark 1.1 in [31], p. 48), where  $\xi$  is a unit normal vector field along  $\varphi$ .

In the case (1) of Theorem 7.1, i.e.,  $(M^{2n+1}, g_\theta) = (S^{2n+1}, g_\theta)$  is a strictly pseudoconvex  $CR$  manifold and  $Jd\varphi(T)$  is tangential, we have

$$0 = h(Jd\varphi(T), \xi) = h(J^2 d\varphi(T), J\xi) = h(d\varphi(T), J(-\xi)),$$

and  $h(d\varphi(T), d\varphi(H(M))) = 0$ . Then, the principal curvature vector field  $J(-\xi)$  with principal curvature  $\lambda_2 = 2 \cot(2u)$  coincides with  $d\varphi(X)$  for some  $X \in H(M)$ . Since

$$\|X\| = \|d\varphi(X)\| = \|J(-\xi)\| = \|\xi\| = 1,$$

we can choose an orthonormal basis  $\{X_i\}_{i=1}^{2n}$  of  $H(M)$  in such a way  $X_1 = X$ . Then,  $\{d\varphi(T), d\varphi(X_2), \dots, d\varphi(X_{2n})\}$  give principal curvature vector fields along  $\varphi$  with principal curvature  $\lambda_1 = \cot u$ . Then,

$$\begin{aligned} \tau_b(\varphi) &= \sum_{i=1}^{2n} B_\varphi(X_i, X_i) = 2 \cot(2u) + (2n-1) \cot u \\ &= 2n \cot u - \tan u. \end{aligned} \quad (8.6)$$

Therefore,  $\varphi$  is *pseudo harmonic* if and only if

$$\tau_b(\varphi) = 0 \quad \Leftrightarrow \quad \tan u = \sqrt{2n}. \quad (8.7)$$

By Theorem 7.1, (1),  $\varphi$  is pseudo biharmonic if and only if

$$\|B_\varphi|_{H(M) \times H(M)}\|^2 = \frac{c(2n+3)}{4} = 2n+3. \quad (8.8)$$

Since the left hand side of (8.8) coincides with

$$\begin{aligned} \|B_\varphi|_{H(M) \times H(M)}\|^2 &= (2 \cot(2u))^2 + (2n-1) \cot^2 u \\ &= (\cot u - \tan u)^2 + (2n-1) \cot^2 u \\ &= 2n \cot^2 u - 2 + \tan^2 u, \end{aligned} \quad (8.9)$$

we have that (8.8) holds if and only if

$$2n \cot^2 u + \tan^2 u = 2n + 5 \quad \Leftrightarrow \quad x^2 - (2n+5)x + 2n = 0, \quad (8.10)$$

where  $x = \tan^2 u$ . Therefore,  $\varphi$  is *pseudo biharmonic* if and only if  $\tan u$  is  $\sqrt{\alpha}$  or  $\sqrt{\beta}$ , where  $\alpha$  and  $\beta$  are positive roots of (8.10).

In the case (2) of Theorem 7.1, i.e.,  $(M^{2n+1}, g_\theta) = (S^{2n+1}, g_\theta)$  is a strictly pseudoconvex  $CR$  manifold, and  $Jd\varphi(T)$  is normal, i.e.,  $Jd\varphi(T) = h(Jd\varphi(T)\xi)\xi$ . Then, we have that

$$0 \neq d\varphi(T) = h(d\varphi(T), J(-\xi)) J(-\xi).$$

And  $J(-\xi)$  is the principal curvature vector field along  $\varphi$  with the principal curvature  $\lambda_2$ , and  $d\varphi(H(M))$  is the space spanned by the principal curvature vectors along  $\varphi$  with the principal curvature  $\lambda_1$

since  $h(d\varphi(H(M)), J(-\xi)) = 0$ . Then the pseudo tension field  $\tau_b(\varphi)$  is given by

$$\tau_b(\varphi) = \sum_{i=1}^{2n} B_\varphi(X_i, X_i) = (2n \cot u) \xi \neq 0, \quad (8.11)$$

so that  $\varphi$  is *not pseudo harmonic*. Due to the case (2) of Theorem 7.1 that  $J d\varphi(T)$  is normal,  $\varphi$  is pseudo biharmonic if and only if

$$\begin{aligned} 2n &= \|B_\varphi|_{H(M) \times H(M)}\|^2 \\ &= m_1 \lambda_1^2 \\ &= 2n \cot^2 u \end{aligned} \quad (8.12)$$

occurs. Thus, we obtain

$$\tan^2 u = 1. \quad (8.13)$$

Therefore, if  $\tan u = 1$  ( $u = \frac{\pi}{4}$ ), then the corresponding isometric immersion  $\varphi : (M^{2n+1}, g_\theta) \rightarrow \mathbb{P}^{n+1}(4)$  is *pseudo biharmonic*, but *not pseudo harmonic*.

**Remark 8.1.** *Let us recall our previous work ([16], [17]) that  $\varphi : (M^{2n+1}, g_\theta) \rightarrow \mathbb{P}^{n+1}(4)$  is biharmonic if and only if*

$$\begin{aligned} \|B_\varphi\|^2 &= \tan^2 u + (2n+1) \cot^2 u - 2 = \frac{n+2}{2} 4 \\ \Leftrightarrow x^2 - 2(n+3)x + 2n+1 &= 0, \quad (x = \tan^2 u). \end{aligned} \quad (8.14)$$

*The equation (8.14) has two positive solutions  $\alpha, \beta$ , and if we put  $\tan u = \sqrt{\alpha}$  or  $\sqrt{\beta}$  ( $0 < u < \frac{\pi}{2}$ ), then  $\varphi : (M^{2n+1}, g_\theta) \rightarrow \mathbb{P}^{n+1}(4)$  is biharmonic, and vice versa. Since the mean curvature is given by (8.4),  $\varphi : (M^{2n+1}, g_\theta) \rightarrow \mathbb{P}^{n+1}(4)$  is harmonic (i.e., minimal) if and only if  $\tan u = \sqrt{2n+1}$  ( $0 < u < \frac{\pi}{2}$ ).*

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